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Article

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The Mappping of the Main Functions and Different Variations of YH-DIE

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Abstract

YH-DIE must have continuity . Given the basic algebraic clusters of homogeneous configurations,we can get the basic three equations:

$$\int_0^{x_i} \frac{G(x_i,s)}{(x_i-s)^{\alpha}}\varphi(s)\,ds=f\left(x_i\right);\ \frac{\partial}{\partial x_i}\left(\frac{\partial_{x_i}G}{\sqrt{1+|\nabla G|^2}}\right)=0;\ i=\sum_{x_i=1}^{\infty}\arccos\ \varphi\left(x_i\right)=f\left(\boxed{Yuh}\right)$$

YH-DIE has become a fusion point and access point in the fields of algebraic geometry and partial differential equations, and its mapping on multidimensional algebraic clusters or manifolds is very special. The minimal surface equation is a special case.

Key words: complete Riemannian manifolds; entire solutions; minimal graphs; differential equat; algebraic variety

Introduction

Throughout the paper, the word *yh-basic* will mean defined on all of the basic form of YH-DIE,and the word *yh-varia* will refers to the variants of the equation (such as:

$$G_i(x,s)=\sum_{x,s=1}^{\infty}\frac{1}{12}\frac{s_2^3}{\sqrt{\varphi}}\left[x^3(1-x)+x(1-x)^3\right]$$

). Taking into account the huge literature on hypersurfaces with prescribed mean curvature, notably on minimal and constant mean curvature (CMC) ones, some choices have to be made, and we decided to exclusively focus on a little of the references considering(just like the parts about another *yh-varia*:

$$G(x)=\frac{-\left(4\varphi+4\varphi^2e^{-2x}\right)}{\left[e^{2x}-\varphi^2e^{-2x}+\left(1-\varphi^2\right)+4\varphi x\right]}$$

).Thus, unfortunately, important works concerning embedded (immersed) hypersurfaces have been omitted, unless they provide new results in the graphical case too. Concerning the prescribed mean curvature problem in Euclidean space, that will be briefly recalled in the next section (if we write), we recommend the survey (46) for an excellent account, that also includes Liouville theorems for more general operators

(We require that $G(x,x)$ is not equal to 0, and $0<\alpha<1$. On this basis, we define the function $\eta(x)$ about x as the mapping of $G(x,f(x))$ on $\varphi(x)$, $\eta(x)$ is the Yuheng Operator, or *YHO* for short. We may also introduce its nature in detail in future articles.)

Manifold Mapping

Ambient manifolds of the type $\mathbb{R}\times M$ leave aside various cases of interest, notably interesting representations of the hyperbolic space $\mathbb{H}^{m+1}$. This and other examples motivated us to consider graphs in more general ambient manifolds $\bar{M}$ ((47; 9; 49)), that foliate topologically as products $\mathbb{R}\times M$ along the flow lines of relevant, in general not parallel, vector fields. A sufficiently large class for our purposes is that of warped products

$$\bar{M} = \mathbb{R} \times_h M, \qquad \bar{g} = \mathrm{d}s^2 + h(s)^2 g, \tag{2.1}$$

for some positive $h \in C^\infty(\mathbb{R})$. Note that $X = h(s)\partial_s$ is a conformal field with geodesic flow lines. Given $v : M \to \mathbb{R}$, we define the graph

$$\Sigma^m = \Big\{ (s, x) \in \mathbb{R} \times M, \;\; s = v(x) \Big\},$$

that we call a geodesic graph.

Further ambient spaces, and define the mapping of $G(x)$ on $\mathbb{H}^{m+1}$

For instance, $\mathbb{H}^{m+1}$ admits the following representations:

(i) $\mathbb{H}^{m+1} = \mathbb{R} \times_{e^s} \mathbb{R}^m$, where the slice of constant s is called a horizontal sphere and is a flattened European The average curvature of the Gilead space in the direction of $-\partial_s$ is 1. Next, change the variable $s = -\log x_0$ in the upper half space model:

$$\mathbb{H}^{m+1} = \Big\{ (x_0, x) \in \mathbb{R} \times \mathbb{R}^m \;\; : \;\; x_0 > 0 \Big\}$$

(ii) $\mathbb{H}^{m+1} = \mathbb{R} \times_{\cosh s} \mathbb{H}^m$, with the slices $\{s = s_0\}$ for constant s_0, called hyperspheres, being totally umbilical hyperbolic spaces of normalized mean curvature $H = \tanh(s_0)$ in the direction of $-\partial_s$, and sectional curvature $-(\cosh s_0)^{-2}$.

(iii) $\mathbb{H}^{m+1} = \mathbb{R} \times_{G(x,s)} \mathbb{H}^m$,which uses the main fuction in *yh-basic* has named *hypersphengyus*. This follows by changing variables $s = -\varphi(x_0)$ in the upper half-space model

$$\mathbb{H}^m = \Big\{ (f(x_0), f(x)) \in \mathbb{R} \times \mathbb{R}^{m-1} \;\; : \;\; s < 0 \Big\}$$

Bootstrap Assumptions

In the following, we shall use $\Gamma^{\alpha,i}$ to denote a string of commutation vector fields where at most i of them are weighted. Because we only commute with two derivatives, we have that $0 \le i \le 2$. The energy associated with $\Gamma^{\alpha,2}$ is allowed to grow like $(1+t)^\delta$.

We shall consider two separate energies, $E_1[\gamma](s)^2$ and $E_2[\gamma](s)^2$. They shall both measure a supremum of energy on the time slices Σ_t and outgoing cones truncated above by Σ_t. We take

$$E_1(s) = \sup_{\substack{0 \le t \le s \\ -1 \le u \le t}} \sum_{|\alpha| \le 2} \|\partial \Gamma^{\alpha,1} \gamma\|_{L^2(\Sigma_t)} + \|\bar{\partial} \Gamma^{\alpha,1} \gamma\|_{L^2(C_u(|u|, 2t-u))},$$

and we take

$$E_2(s) = \sup_{\substack{0 \le t \le s \\ -1 \le u \le t}} \sum_{|\alpha| \le 2} (1+t)^{-\delta} \|\partial \Gamma^{\alpha,2} \gamma\|_{L^2(\Sigma_t)} + (1+t)^{-\delta} \|\bar{\partial} \Gamma^{\alpha,2} \gamma\|_{L^2(C_u(|u|, 2t-u))}.$$

Note that we will usually work with the quantities E_1, E_2 which are the square roots of the energies E_1^2, E_2^2. This is purely a notational convenience.

We shall now take various bootstrap assumptions on γ. The remainder of the proof of the global stability of the plane waves will involve recovering the bootstrap assumptions. Most of these bootstrap assumptions will be recovered easily from the various embedding theorems along with some minor arguments after recovering the bootstrap assumptions for the energy. We have listed all of them to record all of the estimates we shall use in recovering the bootstrap assumptions for the energies $E_1(T)^2$ and $E_2(T)^2$, which are the only steps that require controlling nonlinear terms.

In the following, we fix some large, positive p in terms of δ. More precisely, we pick

$$p \ge \frac{2}{\delta}. \tag{3.2}$$

This p will be used for angular Sobolev embeddings. Indeed, we shall use proposition, which gives us control of the an appropriate mixed Lebesgue space norm in terms of commuting with a single weighted commutation field. The choice of p must be large enough in order to take advantage of the volume of χ_{S_t}

With Γ an arbitrary commutation field and $\Gamma^{\alpha,i}$ an arbitrary string of commutation fields where at most i of them are weighted, we let T be the maximal time such that the following bootstrap assumptions are true:

Bootstrap assumption list

$$E_1(T) \le \epsilon^{\frac{3}{4}}, \tag{3.3}$$

$$E_2(T) \le \epsilon^{\frac{3}{4}}, \tag{3.4}$$

$$\sup_{0 \le t \le T} (1+t+r)^{1-\delta} (1+|u|)^{\frac{1}{2}} |\partial\varphi|(t,r,\omega) \le \epsilon^{\frac{3}{4}}, \tag{3.5}$$

$$\sup_{0 \le t \le T} (1+t+r)^{\frac{3}{2}-\delta} |\bar{\partial}\varphi|(t,r,\omega) \le \epsilon^{\frac{3}{4}}, \tag{3.6}$$

$$\|(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}\varphi\|_{L_t^2[0,T] L_x^2} \le \epsilon^{\frac{3}{4}}, \tag{3.7}$$

$$\|(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}\Gamma^{\alpha,1}\varphi\|_{L_t^2[0,T] L_x^2} \le \epsilon^{\frac{3}{4}}, \tag{3.8}$$

$$\|(1+t)^{-2\delta} (1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}\Gamma^{\alpha,2}\varphi\|_{L_t^2[0,T] L_x^2} \le \epsilon^{\frac{3}{4}}, \tag{3.9}$$

$$\sup_{0 \le t \le T} \|(1+|u|)^{\frac{1}{4}} (1+t+r)^{\frac{1}{2}-\frac{3\delta}{4}} \partial\Gamma\varphi\|_{L^4(\Sigma_t)} \le \epsilon^{\frac{3}{4}}, \tag{3.10}$$

$$\|(1+|u|)^{-\frac{1}{4}-\frac{\delta}{2}} (1+t+r)^{\frac{1}{2}-\frac{3\delta}{2}} \bar{\partial}\Gamma\varphi\|_{L_t^2[0,T] L_x^4} \le \epsilon^{\frac{3}{4}}, \tag{3.11}$$

$$\|\chi_{S_t} r\varphi\|_{L_t^\infty[0,T] L_r^\infty L_\omega^p} \le \epsilon^{\frac{3}{4}}, \tag{3.12}$$

$$\|\chi_{S_t} r\partial\varphi\|_{L_t^\infty[0,T] L_r^\infty L_\omega^p} \le \epsilon^{\frac{3}{4}}, \tag{3.13}$$

$$\|(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}} r\bar{\partial}\varphi\|_{L_t^2[0,T] L_r^2 L_\omega^p} \le \epsilon^{\frac{3}{4}}, \tag{3.14}$$

$$\|(1+t)^{-\delta} r\partial\Gamma\varphi\|_{L_t^\infty[0,T] L_r^2 L_\omega^p} \le \epsilon^{\frac{3}{4}}, \tag{3.15}$$

$$\|(1+t)^{-2\delta} (1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}} r\bar{\partial}\Gamma\varphi\|_{L_t^2[0,T] L_r^2 L_\omega^p} \le \epsilon^{\frac{3}{4}}, \tag{3.16}$$

$$\sup_{0 \le t \le T} (1+t+r)^{1-\delta} \chi_{S_t} |\varphi|(t,r,\omega) \le \epsilon^{\frac{3}{4}}, \tag{3.17}$$

$$\sup_{0 \le t \le T} \|(1+t)^{-\delta} \chi_{S_t} \Gamma^{\alpha,2}\varphi\|_{L^2(\Sigma_t)} \le \epsilon^{\frac{3}{4}}. \tag{3.18}$$

$$\|(1+t)^{-\delta} \chi_{S_t} r\Gamma\varphi\|_{L_t^\infty[0,T] L_r^\infty L_\omega^p} \le \epsilon^{\frac{3}{4}}, \tag{3.19}$$

Continuation of discussion regarding bootstrap assumptions

We shall improve the bounds from $\epsilon^{\frac{3}{4}}$ to being $C\epsilon$. This will recover the bootstrap assumptions when ϵ is sufficiently small, giving us the desired result.

The main remaining difficulty in establishing Theorem is recovering the bootstrap assumptions on the energy. We have the following proposition.

Proposition 3.1. *The bootstrap assumptions in Section* **??** *imply that* $E_1(T) \le C\epsilon$ *and* $E_2(T) \le C\epsilon$.

We shall now assume that we have shown Proposition 3.1. These estimates are established. We shall show that, as a result of this, we can recover all of the other bootstrap assumptions. For each of these, the portion of the estimates where $t < 10$ follow from the usual Sobolev embedding on a spacetime cube of side length 100, so we will only worry about the parts of these norms that have $t > 2$.

We shall first use Proposition to recover the pointwise bootstrap assumptions on $\partial\gamma$ and $\bar{\partial}\gamma$. They are direct consequences of the Klainerman-Sobolev inequalities we have established.

Lemma 3.2. *Assuming Proposition 3.1, we recover bootstrap assumptions* (3.5) *and* (3.6) *and improve the bound to* $C\epsilon$. *Namely, we derive the estimates*

1. $(1+t+r)^{1-\delta}(1+|u|)^{\frac{1}{2}}|\partial\gamma|(t,r,\omega) \le CE_2(t)$,
2. $(1+t+r)^{\frac{3}{2}-\delta}|\bar{\partial}\gamma|(t,r,\omega) \le CE_2(t)$.

Proof. The first estimate follows from applying the proposition and noting that the right hand side is controlled by E_2 recovered in Proposition 3.1. The second estimate similarly follows similarly using Proposition instead when $|u| < \frac{t}{100}$ and follows directly from the first estimate when $|u| \ge \frac{t}{100}$.

We shall now recover the spacetime L^2 estimates on the good derivatives.

Lemma 3.3. *Assuming Proposition 3.1, we recover bootstrap assumptions* (3.7), (3.8) , (3.9)*, namely we have that*

1. $\|(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}}\bar{\partial}\gamma\|_{L_t^2[2,T]L_x^2} \le C\epsilon$,
2. $\|(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}}\bar{\partial}\Gamma^{\alpha,1}\gamma\|_{L_t^2[2,T]L_x^2} \le C\epsilon$,
3. $\|(1+t)^{-2\delta}(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}}\bar{\partial}\Gamma^{\alpha,2}\gamma\|_{L_t^2[2,T]L_x^2} \le C\epsilon$.

Proof. The first two parts follow immediately from the fact that $E_1(T)^2$ controls the characteristic energy along with the fact that $(1+|u|)^{-1-\delta}$ is integrable in u. We now turn to the third part.

We begin by noting that, as a consequence of Proposition 3.1, we have that

$$\int_{|u|}^{2s-u}\int_{S^2}(\bar{\partial}\Gamma^{\alpha,2}\gamma)^2 r^2\, d\omega dv \le C\epsilon^2(1+s)^{2\delta} \tag{3.20}$$

for every $s \le T$, where we are integrating on the cone C_u. Thus, with $h(u,v) = \int_{S^2}(\bar{\partial}\Gamma^{\alpha,2}\gamma)^2 r^2 d\omega$ and integrating by parts in v, we have that

$$\int_{|u|}^{2T-u}(1+t)^{-4\delta}h\, dv = (1+T)^{-4\delta}\int_{|u|}^{2T-u}h(u,v')dv'$$
$$+2\delta\int_{|u|}^{2T-u}\left(1+\frac{1}{2}(v+u)\right)^{-1-4\delta}\int_{|u|}^{v}h(u,v')dv'dv. \tag{3.21}$$

Now, we note that

$$\int_{|u|}^{v}h(u,v')dv' \le C\epsilon^2(1+t)^{2\delta} = C\epsilon^2\left(1+\frac{1}{2}(v+u)\right)^{2\delta}.$$

Thus, we have that

$$\delta\int_{|u|}^{2T-u}\left(1+\frac{1}{2}(v+u)\right)^{-1-4\delta}\int_{|u|}^{v}h(u,v')dv'dv \le C\delta\epsilon^2\int_{|u|}^{2T-u}\left(1+\frac{1}{2}(v+u)\right)^{-1-2\delta}dv \le C\epsilon$$

and similarly for the first term in (3.21). Multiplying (3.21) by $(1+|u|)^{-1-\delta}$, integrating in u, we get that

$$\begin{aligned}\|(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}}(1+t)^{-2\delta}\bar{\partial}\Gamma^{\alpha,2}\gamma\|_{L_t^2L_x^2}^2 &\le 4\int_{-1}^{T}(1+|u|)^{-1-\delta}\int_{|u|}^{2t-u}(1+t)^{-4\delta}\int_{S^2}(\bar{\partial}\Gamma^{\alpha,2}\gamma)^2r^2d\omega\, dv\, du\\ &\le 4\int_{-1}^{T}(1+|u|)^{-1-\delta}\int_{|u|}^{2t-u}(1+t)^{-4\delta}h\, dv\, du\\ &\le C\epsilon^2,\end{aligned}$$

where we have used the fact that $(1+|u|)^{-1-\delta}$ is integrable in u. This gives us the desired result.

We now recover the $L_t^\infty L_x^4$ estimates on all derivatives and the $L_t^2L_x^4$ estimates on good derivatives. They are both a consequence of the L_x^6 Klainerman-Sobolev inequality and an interpolation argument.

Commutative Diagrams

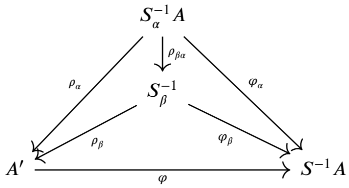
Consider an incremental filter phase family (S_α) of the multiplicative subset of A (we stipulate that $\alpha \le \beta$ is $S_\alpha \subseteq S_\beta$, and set S is the multiplicative subset $\bigcup_\alpha S_\alpha$. For $\alpha \le \beta$, we let $\rho_{\beta\alpha} = \rho_A^{S_\alpha,S_\beta}$,and according to

$$\rho^{U,S} = \rho^{U,T}\circ\rho^{T,S} \tag{4.22}$$

(S,T,U are the three multiplicative subsets of A , and the inductive system of the ring formed by $S \subseteq T \subseteq U$. See (24; 8)), so that an inductive limit cycle A' can be defined.

Change the multiplicative subset and make φ an isomorphism

Let ρ_α be a canonical homomorphism $S_\alpha^{-1}A \to A'$, and let $\varphi_\alpha = \rho_A^{S,S_\alpha}$, so according to (3), for $\alpha \le \beta$, we can always have $\varphi_\alpha = \varphi_\beta \bigcirc \rho_{\beta\alpha}$, and we can uniquely define a homomorphic $\varphi : A' \to S^{-1}A$, making the diagrams



So we can uniquely define a homomorphic $\varphi : A' \to S^{-1}A$, so that the graphs (where $\alpha \le \beta$) are exchanged. This φ is an isomorphism. In fact, by the way of constructing φ you can immediately know that it is full.

On the other hand, if $\rho_\alpha\left(a/s_\alpha\right) \in A'$ satisfies $\varphi\left(\rho_\alpha\left(a/s_\alpha\right)\right) = 0$, then there is $a/s_\alpha = 0$ in $S^{-1}A$. In other words, you can find $s \in S$ makes $sa = 0$. However, you can find $\beta \geqslant \alpha$ makes $s \in S_(\beta)$, so $\rho_\alpha\left(a/s_\alpha\right) = \rho_\beta\left(sa/ss_\alpha\right) = 0$ It can be seen that φ is single. The same method also applies to A modulates M, which defines the following canonical isomorphism

$$\varinjlim S_\alpha^{-1}A \xrightarrow{\sim} \left(\varinjlim S_\alpha\right)^{-1}A,$$

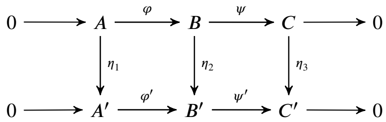
$$\varinjlim S_\alpha^{-1}M \xrightarrow{\sim} \left(\varinjlim S_\alpha\right)^{-1}M$$

Definition 4.1. *Suppose* $\mathbb{L}_1, \mathbb{L}_2$ *are quasigroups. Then a triple* (α,β,γ) *of maps from* $\mathbb{L}_1$ *to* $\mathbb{L}_2$ *is a* homotopy *from* $\mathbb{L}_1$ *to* $\mathbb{L}_2$ *if for any* $p, q \in \mathbb{L}_1$,

$$\alpha(p)\,\beta(q) = \gamma(pq)\,. \tag{4.23}$$

If (α,α,α) *is a homotopy, then* α *is a* quasigroup homomorphism. *If each of the maps* α,β,γ *is a bijection, then* (α,β,γ) *is an* isotopy. *An isotopy from a quasigroup to itself is an* autotopy. *The set of all autotopies of a quasigroup* $\mathbb{L}$ *is clearly a group under composition. If* (α,α,α) *is an autotopy, then* α *is an automorphism of* $\mathbb{L}$*, and the group of automorphisms is denoted by* Aut $(\mathbb{L})$.

The second isomorphism is still functoric for M.



Properties of Base-$\Theta^{\pm}$-bridges, Base-Θ^{ell}-bridges of YH-DIE

Relative to a fixed collection ofinitial Θ-data:

(i) The set of isomorphisms between two $\mathcal{D}-\Theta^{\pm}$ -bridges forms a torsor over the group

$$\{\pm 1\} \times \left(\{\pm 1\}^{\mathrm{V}}\right)$$

- where the first (respectively, second) factor corresponds to poly-automorphisms of the sort described in (15) Example 6.2, (ii) (respectively, Example 6.2, (iii)). Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of $\mathbb{F}_l^{\pm}$ -groups between the index sets of the capsules involved.

(ii) The set of isomorphisms between two $\mathcal{D}-\Theta^{\text{ell}}$ -bridges forms an $\mathbb{F}_l^{\times\pm}$ torsor - i.e., more precisely, a torsor over a finite group that is equipped with a natural outer isomorphism to $\mathbb{F}_l^{\times\pm}$. Moreover, this set of isomorphisms maps bijectively, by considering the induced bijections, to the set of isomorphisms of $\mathbb{F}_l^{\pm}$ -torsors between the index sets of the capsules involved.

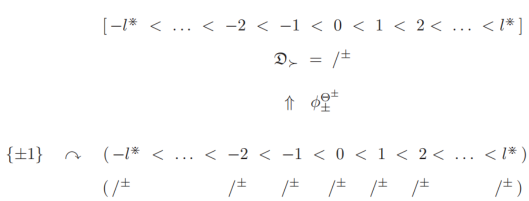
(iii) The set of isomorphisms between two $\mathcal{D}-\Theta^{\pm\text{ell}}$ -Hodge theaters forms $a\{\pm 1\}$ -torsor. Moreover, this set of isomorphisms maps bijectively, by considering the induced bijections, to the set of isomorphisms of $\mathbb{F}_l^{\pm}$ -groups between the index sets of the capsules involved.

(iv) Given a $\mathcal{D}-\Theta^{\pm}$ -bridge and a $\mathcal{D}-\Theta^{\text{ell}}$ -bridge, the set of capsule-t-full polyisomorphisms between the respective capsules of $\mathcal{D}$ -prime-strips which allow one to glue the given $\mathcal{D}-\Theta^{\pm}$ - and $\mathcal{D}-\Theta^{\text{ell}}$ -bridges together to form a $\mathcal{D}-\Theta^{\pm\text{ell}}$ - Hodge theater forms a torsor over the graph

$$\mathbb{F}_l^{\times\pm} \times \left(\{\pm 1\}^{\mathrm{V}}\right)$$

- where the first factor corresponds to the $\mathbb{F}_l^{\times\pm}$ of (ii); the subgroup $\{\pm 1\} \times .(\{\pm 1\}^{\mathrm{V}})$ corresponds to the group of (i). Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of $\mathbb{F}_l^{\pm}$ -torsors between the index sets of the capsules involved.

(v) Given a $\mathcal{D}-\Theta^{\text{ell}}$ -bridge [simple -cf. the discussion of Example 6.2,(i)] functorial algorithm for constructing, up to an $\mathbb{F}_l^{\times\pm}$ indeterminacy [cf. $(ii),(iv)$], from the given $\mathcal{D}-\Theta^{\text{ell}}$ - bridge a $\mathcal{D}-\Theta^{\pm\text{ell}}$ -Hodge theater whose underlying $\mathcal{D}-\Theta^{\text{ell}}$ -bridge is the given $\mathcal{D}-\Theta^{\text{ell}}$ -bridge.



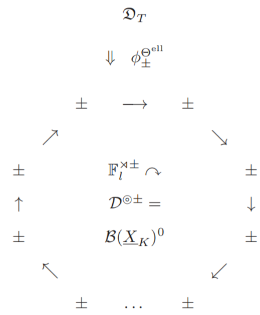


Fig. 1. This is the combinatorial structure of a $\mathcal{D}-\Theta^{\text{ell}}$ -Hodge theater of φ,one of the main functions of YH-DIE.Learn more in (15; 13; 59; 63)

Different Variations

Suppose we have the equation (we just call equations like it *yh-varia-basic-PDEs*, or *yh-vbps* for short)

$$f_{xx} + f_{yy} + x f_y = 0 \tag{5.24}$$

and we would like to transform the equation from the {x, y} variables to the {u, v} variables, where

$$u = x, \quad v = \frac{x}{y} \tag{5.25}$$

Note that the inverse transformation is given by $x = u, y = u/v$. We define $g(u, v)$ to be equal to the function $f(x, y)$ when written in the new variables. That is,

$$f(x, y) = g(u, v) = g\left(x, \frac{x}{y}\right)$$

Now we create the needed derivative terms, carefully applying the chain rule. For example, by differentiating equation 5.25 with respect to x , we obtain

$$f_x(x, y) = g_u \frac{\partial}{\partial x}(u) + g_v \frac{\partial}{\partial x}(v) = g_1 \frac{\partial}{\partial x}(x) + g_2 \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$
$$= g_1 + g_2 \frac{1}{y} = g_1 + \frac{v}{u} g_2$$

where we have used a subscript of "1" (" 2 ") to indicate a derivative with respect to the first (second) argument of the function $g(u, v)(i.e., g_1(u, v) = g_u(u, v))$. Use of this "slot notation" tends to minimize errors. In a like manner, we find

$$f_y(x, y) = g_u \frac{\partial}{\partial y}(u) + g_v \frac{\partial}{\partial y}(v) = g_1 \frac{\partial}{\partial y}(x) + g_2 \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$
$$= -\frac{x}{y^2} g_2 = -\frac{v^2}{u} g_2$$

Transforming partial dierential *yh-basic* equations

The second order derivatives can be calculated similarly:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(f_x(x, y)\right) = \frac{\partial}{\partial x}\left(g_1 + \frac{1}{y} g_2\right)$$
$$= g_{11} + \frac{2v}{u} g_{12} + \frac{v^2}{u^2} g_{22}$$
$$f_{xy}(x, y) = \frac{\partial}{\partial x}\left(-\frac{x}{y^2} g_2\right) = -\frac{u^2}{v^2} g_2 - \frac{u^3}{v^3} g_{12} - \frac{u^2}{v^2} g_{22}$$
$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(-\frac{x}{y^2} g_2\right) = \frac{2v^3}{u^2} g_2 + \frac{v^4}{u^2} g_{22}$$

Finally, then, we can determine what equation 5.24 looks like in the new variables:

$$\begin{aligned} 0 &= f_{xx} + f_{yy} + x f_y \\ &= \left(g_{11} + \frac{2v}{u} g_{12} + \frac{v^2}{u^2} g_{22}\right) + \left(\frac{2v^3}{u^2} g_2 + \frac{v^4}{u^2} g_{22}\right) + (u) \\ &= \frac{v^2\left(2v - u^2\right)}{u^2} g_v + g_{uu} + \frac{2v}{u} g_{uv} + \frac{v^2\left(1+v^2\right)}{u^2} g_{vv} \end{aligned}$$

Euler Transformation of *yh-varia* and transform into G(x)

Then,let's see the transformations of *yh-varia*.

Given the first order partial differential equation in two independent variables, $\varphi(x, y, z, p, q) = 0$ (with, as usual, $p = z_x, q = z_y$ and $z_{xx}/0$ the transformation

$$\left\{\begin{array}{l} x = Zx \\ y = Y \\ z = XZ_X - Z \\ p = X \\ q = -Z_Y \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} X = z_x \\ Y = y \\ Z = xz_x - z \\ P = x \\ Q = -z_y \end{array}\right\}$$

is known as the Euler transformation. Note that $Z_Y + z_y = 0$. Under this transformation, the original equation transforms into $\varphi\left(Z_X, Y, XZ_X - .\ Z, X, -Z_Y\right) = 0$.

As an example, the equation $G(xp - z, y, p, q) = 0$ becomes, under the Euler transformation, $G\left(Z, Y, X, -Z_Y\right) = 0$. As another example, the Clairaut partial differential equation $\varphi = z - \left(xz_x + yz_y + f\left(z_x, z_y\right)\right) = 0$ is transformed into $F = Z - YZ_Y + f\left(X, -Z_Y\right) = 0$. Note that this latter equation is really an ordinary differential equation for $Z = Z(Y)$ (the variable X acts as a parameter).

(In next paper, we will also talk about the same topic of YH-DIE. This paper mean talks the diffirent variations of basic partial differential equations of YH-DIE. Because of the space, integral equations and heat conduction partial differential equations will be written in the next paper.)

Gradient Estimates in $\mathbb{R} \times M$

As we saw above, under general assumptions on b, f, ℓ solutions of

$$\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = b(x) f(u) \ell(|Du|) \qquad \text{on } M$$

indeed satisfy $f(u) \equiv 0$, in particular

$$\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \qquad \text{on } M. \tag{6.26}$$

However, to infer the constancy of u solving (6.26), one needs rather different arguments (and more binding assumptions) than those leading to the results in the previous section; indeed, while the above theorems apply to differential *inequalities*, those in this section are very specific to the equality case, unless in the special situation when M has slow volume growth. This consideration is not surprising, as it parallels the case of harmonic functions: positive solutions of $\Delta u = 0$ are constant on each complete manifold with $\mathrm{Ric} \geq 0$ by (58; 50), while the constancy of every positive solution of $\Delta u \leq 0$ is equivalent to the parabolicity of M (cf. (62; 60)) that, for complete manifolds with $\mathrm{Ric} \geq 0$, is equivalent to the slow volume growth condition

$$\int^{\infty} \frac{s\mathrm{d}s}{|B_s|} = \infty. \tag{6.27}$$

As a matter of fact, (6.27) is sufficient for the parabolicity of a complete manifold M, regardless to any curvature condition (cf. (60, Thm. 7.5)).

Ricci curvature and gradient estimate for minimal graphs

Our approach in (7) to obtain (47) Theorem 1.1 still relies on a gradient estimate. Precisely, we prove

Theorem 6.1 ((7))**.** *Let M be a complete manifold of dimension $m \geq 2$ satisfying*

$$\mathrm{Ric} \geq -(m-1)\kappa^2, \tag{6.28}$$

for some constant $\kappa \geq 0$. Let $\Omega \subset M$ be an open subset and let $u \in C^\infty(\Omega)$ be a positive solution of (6.26) *on* Ω*. If either*

(*i*) *Ω has locally finite perimeter and*

$$\liminf_{r\to\infty} \frac{\log|\partial\Omega \cap B_r|}{r^2} < \infty, \qquad \textit{or}$$

(*ii*) *$u \in C(\overline{\Omega})$ and is constant on $\partial\Omega$.*

Then

$$\frac{\sqrt{1+|Du|^2}}{e^{\kappa u\sqrt{m-1}}} \leq \max\left\{1, \limsup_{x\to\partial\Omega} \frac{\sqrt{1+|Du(x)|^2}}{e^{\kappa u(x)\sqrt{m-1}}}\right\} \qquad \textit{on } \Omega. \tag{6.29}$$

In the particular case $\Omega = M$,

$$\sqrt{1+|Du|^2} \leq e^{\kappa u\sqrt{m-1}} \qquad \textit{on } M. \tag{6.30}$$

If equality holds in (6.30) *at some point, then $\kappa = 0$ and u is constant.*

Despite we found no explicit example, we feel likely that the bound (6.30) be sharp also for $\kappa > 0$, in the sense that the constant $\kappa\sqrt{m-1}$ cannot be improved. Our estimate should be compared to the one for positive harmonic functions on manifolds satisfying (6.28), obtained by P. Li and J. Wang (18) by refining Cheng-Yau's argument:

$$|Du| \leq (m-1)\kappa u \qquad \text{on } M, \tag{6.31}$$

and its version for sets with boundary in [65 Thm. 2.24]:

$$\frac{|Du|}{(m-1)\kappa u} \leq \max\left\{1, \limsup_{x\to\partial\Omega} \frac{|Du(x)|}{(m-1)\kappa u(x)}\right\} \qquad \text{on } \Omega. \tag{6.32}$$

Ricci curvature and gradient estimate for CMC graphs

The above method allows for generalizations to the CMC case, obtained in the very recent (56), that apply to the rigidity of capillary graphs over unbounded regions. We detail the application in the next section. Although the guiding idea is the same as the one for Theorem 6.1, the difficulty to deal with nonzero H makes the statement of the next result, and its proof, more involved.

Theorem 6.2 ((56))**.** *Let M be a complete manifold of dimension $m \geq 2$ satisfying*

$$\mathrm{Ric} \geq -(m-1)\kappa^2 \qquad on \;\; M$$

for some constant $\kappa \geq 0$. Let $\Omega \subseteq M$ be an open subset and let $u \in C^\infty(\Omega)$ be a positive solution of

$$\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = mH \qquad on \;\; \Omega,$$

for some $H \in \mathbb{R}$. Suppose that either (i) or (ii) of Theorem 6.1 are satisfied. Let $C \geq 0$ and $A \geq 1$ satisfy

- *if $H < 0$,*

$$mH^2 + C^2 - (m-1)\kappa^2 \geq 0 \tag{6.33}$$

- *if $H \geq 0$,*

$$mH^2 - \frac{CmH}{A} + (C^2 - (m-1)\kappa^2)\frac{A^2-1}{A^2} \geq 0, \tag{6.34}$$

$$mHCA + 2(C^2 - (m-1)\kappa^2) \geq 0. \tag{6.35}$$

Then,

$$\frac{\sqrt{1+|Du|^2}}{e^{Cu}} \leq \max\left\{A, \limsup_{y\to\partial\Omega}\frac{\sqrt{1+|Du(y)|^2}}{e^{Cu(y)}}\right\} \qquad on \;\; \Omega. \tag{6.36}$$

In particular, if $\Omega = M$,

$$\sqrt{1+|Du|^2} \leq Ae^{Cu} \qquad on \;\; M.$$

Simplicial Spanning Trees

Let $G = (V, E)$ be a graph with vertex set V and edge set E and for a vertex $v \in V$, denote by $\deg(v)$ its degree. G is called a k-regular graph, if $\deg(v) = k$ for all $v \in V$. A sub-graph $T = (V', E')$ of G is called a *spanning tree* of G if T is an acyclic, connected graph such that $V' = V$. For a graph G, denote by $\kappa_1(G)$ the number of spanning trees in it.

A classical model for random k-regular graphs, called the random matching model $\mathcal{G}_{n,k}$, is defined for $k \geq 1$ and $n \in \mathbb{N}$ even as the graph with vertex set $[n] := \{1, 2, \dots, n\}$ and edge set which is the union of k independent and uniformly distributed perfect matching on the set $[n]$.

Weak convergence of the empirical spectral distributions

Definition 7.1. *Let X be a d-dimensional simplicial complex. The* oriented line-graph *of X, denoted $\vec{G}_d(X) = (X_\pm^{d-1}, \vec{E}_d(X))$, is the graph whose vertex set $X_\pm^{d-1}$ is composed of all oriented $(d-1)$-faces in X and its edge set $\vec{E}_d(X)$ is defined to be the set of pairs $\{\sigma, \sigma'\}$ from $X_\pm^{d-1}$ such that σ is a neighbour of σ' in X.*

Proposition 7.1. *Let X be a pure, d-complex such that $\deg(\sigma) < \infty$ for all $\sigma \in X^{d-1}$. For $r \geq 0$ and $\sigma, \sigma' \in X_\pm^{d-1}$ denote by $\phi_r(X; \sigma, \sigma')$ the number of paths of length r in $\vec{G}_d(X)$ from σ to σ'. Then*

$$\langle A_X^r \mathbf{1}_\sigma, \mathbf{1}_{\sigma'}\rangle = \phi_r(X; \sigma, \sigma') - \phi_r(X; \sigma, \overline{\sigma'}), \qquad \forall \sigma, \sigma' \in X_\pm^{d-1}.$$

In particular, for every choice of orientation X_+^{d-1} for each of the $(d-1)$-faces

$$\frac{1}{|X_+^{d-1}|}\mathrm{tr}(A_{X_i}^r) = \frac{1}{|X_+^{d-1}|}\sum_{\sigma\in X_+^{d-1}}\langle A_X^r \mathbf{1}_\sigma, \mathbf{1}_\sigma\rangle$$

$$= \frac{1}{|X_+^{d-1}|}\sum_{\sigma\in X_+^{d-1}}\big(\phi_r(X;\sigma,\sigma) - \phi_r(X;\sigma,\overline{\sigma})\big).$$

Proof. The proof follows by induction on r. For $r = 0$, $A_X^0 = \mathrm{Id}_{\Omega^{d-1}(X)}$ and thus

$$\langle A_X^0 \mathbf{1}_\sigma, \mathbf{1}_{\sigma'}\rangle = \begin{cases} 1 & \sigma = \sigma' \\ -1 & \sigma = \overline{\sigma'} \\ 0 & \text{otherwise}\end{cases}.$$

On the other hand, from the definition of ϕ_r, we have that

$$\phi_0(X; \sigma, \sigma') = \begin{cases} 1 & \sigma = \sigma' \\ 0 & \text{otherwise}\end{cases},$$

and thus

$$\phi_0(X; \sigma, \sigma') - \phi_0(X; \sigma, \overline{\sigma'}) = \begin{cases} 1 & \sigma = \sigma' \\ -1 & \sigma = \overline{\sigma'} \\ 0 & \text{otherwise}\end{cases},$$

which proves the result for $r = 0$.

Turning to the induction step, assume the result holds for r and observe $\langle A_X^{r+1}\mathbf{1}_\sigma, \mathbf{1}_{\sigma'}\rangle$. By definition $\langle A_X^{r+1}\mathbf{1}_\sigma, \mathbf{1}_{\sigma'}\rangle = \langle A_X^r(A_X\mathbf{1}_\sigma), \mathbf{1}_{\sigma'}\rangle$ and since

$$A_X\mathbf{1}_\sigma(\sigma'') = \sum_{\rho\sim\sigma''}\mathbf{1}_\sigma(\rho) = \begin{cases} 1 & \sigma'' \sim \sigma \\ -1 & \sigma'' \sim \overline{\sigma} \\ 0 & \text{otherwise}\end{cases},$$

namely

$$A_X\mathbf{1}_\sigma = \sum_{\rho\sim\sigma}\mathbf{1}_\rho,$$

we conclude that

$$\langle A_X^{r+1}\mathbf{1}_\sigma, \mathbf{1}_{\sigma'}\rangle = \Big\langle A_X^r\Big(\sum_{\rho\sim\sigma}\mathbf{1}_\rho\Big), \mathbf{1}_{\sigma'}\Big\rangle = \sum_{\rho\sim\sigma}\langle A_X^r\mathbf{1}_\rho, \mathbf{1}_{\sigma'}\rangle.$$

Thus by induction

$$\langle A_X^{r+1}\mathbf{1}_\sigma, \mathbf{1}_{\sigma'}\rangle = \sum_{\rho\sim\sigma}\big(\phi_r(X;\rho,\sigma') - \phi_r(X;\rho,\overline{\sigma'})\big) = \phi_{r+1}(X;\sigma,\sigma') - \phi_{r+1}(X;\sigma,\overline{\sigma'}),$$

where in the last step we used the fact that any path of length $r+1$ from σ to σ' is composed of one step from σ to a neighbour ρ of σ in $\vec{G}_d(X)$ followed by a path of length r from ρ to σ'.

The formula for $\langle \mu_{A_X}, x^r\rangle$ follows from the fact that $(\mathbf{1}_\sigma)_{\sigma\in X_+^{d-1}}$ is an orthonormal basis for $\Omega^{d-1}(X)$.

The asymptotic number of simplicial spanning trees

Definition 7.2. *Chebyshev polynomials of the first kind are defined as the unique sequence of polynomials $(T_n)_{n=0}^\infty$ satisfying $\deg(T_n) = n$ for all $n \in \mathbb{N}_0$ and $T_n \circ \cos(x) = \cos(nx)$ for all $n \in \mathbb{N}_0$.*

Chebyshev's polynomials are classical and well-studied, c.f. (42). Below we collect several useful properties they posses.

Orthogonality

$$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx=\begin{cases}0 & \text{if } n\neq m\\ \pi & \text{if } n,m=0\\ \frac{\pi}{2} & \text{if } n=m\neq 0\end{cases}. \tag{7.37}$$

Logarithmic generating function For all $|t|<1$ and $x\in\mathbb{R}$

$$\log(1-2xt+t^2)=-2\sum_{n=1}^{\infty}T_n(x)\cdot\frac{t^n}{n}. \tag{7.38}$$

Expansion of powers via Chebyshev's polynomials For every $n\geq 0$

$$x^{2n+1}=2^{-2n}\sum_{m=0}^{n}\binom{2n+1}{n-m}T_{2m+1}(x) \tag{7.39}$$

and

$$x^{2n}=2^{1-2n}\sum_{m=1}^{n}\binom{2n}{n-m}T_{2m}(x)+2^{-2n}\binom{2n}{n}. \tag{7.40}$$

Hence for every converging power series

$$\begin{aligned}\sum_{n=0}^{\infty}c_nx^n=&\left(\sum_{n=0}^{\infty}\frac{c_{2n}}{2^{2n}}\binom{2n}{n}\right)T_0(x)\\ &+\sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}\frac{c_{2n+1}}{2^{2n}}\binom{2n+1}{n-m}\right)T_{2m+1}(x)\\ &+\sum_{m=1}^{\infty}2\left(\sum_{n=m}^{\infty}\frac{c_{2n}}{2^{2n}}\binom{2n}{n-m}\right)T_{2m}(x).\end{aligned} \tag{7.41}$$

Let $g:[-1,1]\to R$ be a continuous function. Denoting $h(x)=g(x)\sqrt{1-x^2}$, the orthogonality property (7.37) with respect to the function $(1-x^2)^{-1/2}$, enables us to develop h as a power series in Chebyshev polynomials

$$h(x)=\sum_{n=0}^{\infty}\alpha_nT_n(x),$$

where α_n is given by

$$\alpha_n=\begin{cases}\frac{1}{\pi}\int_{-1}^{1}T_0(x)g(x) & ,\text{if } n=0\\ \\ \frac{2}{\pi}\int_{-1}^{1}T_n(x)g(x)dx & ,\text{if } n\geq 1\end{cases}.$$

In particular, if the Chebyshev power series of h converges uniformly on $(-1,1)$ we get from (7.38) and integration term by term that

$$\begin{aligned}\int_{-1}^{1}\log(1-2xt+t^2)\cdot g(x)dx&=\int_{-1}^{1}\left(-2\sum_{n=1}^{\infty}T_n(x)\frac{t^n}{n}\right)\cdot\frac{h(x)}{\sqrt{1-x^2}}dx\\ &=-2\sum_{n=1}^{\infty}\frac{t^n}{n}\int_{-1}^{1}\frac{T_n(x)h(x)}{\sqrt{1-x^2}}dx=-\pi\sum_{n=1}^{\infty}\frac{\alpha_n}{n}t^n.\end{aligned} \tag{7.42}$$

Pseudoautomorphisms

Suppose now $\mathbb{L}$ is a loop and (α,β,γ) is an autotopy of $\mathbb{L}$. Let $B=\alpha(1)$, $A=\beta(1)$, $C=\gamma(1)$. It is clear that $BA=C$. Moreover, from (4.23) we see that

$$\begin{aligned}\alpha(p)&=\gamma(p)/A\\ \beta(p)&=B\backslash\gamma(p).\end{aligned}$$

We can rewrite (4.23) as

$$\alpha(p)\cdot B\backslash\alpha(q)A=\alpha(pq)A$$

If $B=1$, then, we obtain a *right pseudoautomorphism* α *of* $\mathbb{L}$ *with companion* A, which we'll denote by the pair (α,A), and which satisfies

$$\alpha(p)\cdot\alpha(q)A=\alpha(pq)A. \tag{8.43}$$

We have the following useful relations for quotients:

$$\alpha(q\backslash p)A=\alpha(q)\backslash\alpha(p)A \tag{8.44a}$$

$$\alpha(p/q)\cdot\alpha(q)A=\alpha(p)A \tag{8.44b}$$

There are several equivalent ways of characterizing *right pseudoautomorphisms*.

Theorem 8.1. *Let* $\mathbb{L}$ *be a loop and suppose* $\alpha:\mathbb{L}\longrightarrow\mathbb{L}$. *Also, let* $A\in\mathbb{L}$ *and* $\gamma=R_A\circ\alpha$. *Then the following are equivalent:*

1. (α,A) *is a* right pseudoautomorphism of $\mathbb{L}$ with companion A.
2. (α,β,γ) *is an autotopy of* $\mathbb{L}$ *with* $\alpha(1)=1$ *and* $\beta(1)=\gamma(1)=A$.
3. $\gamma(1)=A$ *and* γ *satisfies*

$$\gamma(p)\gamma\left(q\gamma^{-1}(1)\right)=\gamma(pq). \tag{8.45}$$

Remark 8.2. *Similarly, if* $A=1$, *then we can rewrite (4.23) as*

$$B\beta(p)\cdot\beta(q)=B\beta(pq)$$

and in this case, β *is a* left pseudoautomorphism *with companion* B. *Finally, suppose* $C=1$, *so that then* $A=B^{\rho}$, *and we can rewrite (4.23)*

$$\gamma(p)/B^{\rho}\cdot B\backslash\gamma(q)=\gamma(pq)$$

so that in this case, γ *is a* middle pseudoautomorphism *with companion* B.

Example 8.3. *In a Moufang loop, consider the map* Ad_q, *given by* $p\longmapsto qpq^{-1}$. *Note that this can be written unambiguously due to diassociativity. Then, this is a right pseudoautomorphism with companion* q^3 *(28, Lemma 1.2). Indeed, using diassociativity for* $\{q,xy\}$, *we have*

$$q(xy)q^{-1}\cdot q^3=q(xy)q^2.$$

On the other hand,

$$\begin{aligned}qxq^{-1}\cdot qyq^2&=q\left(xq^{-1}\right)\cdot(qyq)q\\ &=\left(q\left(xq^{-1}\cdot qyq\right)\right)q\\ &=(q(xy\cdot q))q\\ &=q(xy)q^2,\end{aligned}$$

where we have use appropriate Moufang identities. Hence, indeed,

$$q(xy)q^{-1}\cdot q^3=\left(qxq^{-1}\right)\left(qyq^{-1}\cdot q^3\right).$$

In general, the adjoint map on a loop is not *a pseudoautomorphism or a loop homomorphism. For each* $q\in\mathbb{L}$, Ad_q *is just a bijection that preserves* $1\in\mathbb{L}$.

However, as we see above, it is a pseudoautomorphism if the loop is Moufang. Keeping the same terminology as for groups, we'll say that Ad *defines an adjoint action of* $\mathbb{L}$ *on itself, although for a non-associative loop, this is not an action in the usual sense of a group action.*

We can easily see that the right pseudoautomorphisms of $\mathbb{L}$ form a group under composition. Denote this group by $\mathrm{PsAut}^R(\mathbb{L})$. Clearly, $\mathrm{Aut}(\mathbb{L}) \subset \mathrm{PsAut}^R(\mathbb{L})$. Similarly for left and middle pseudoautomorphisms. More precisely, $\alpha \in \mathrm{PsAut}^R(\mathbb{L})$ if there exists $A \in \mathbb{L}$ such that (8.43) holds. Here we are not fixing the companion. On the other hand, consider the set $\Psi^R(\mathbb{L})$ of all pairs (α, A) of *right pseudoautomorphisms with fixed companions*. This then also forms a group.

Lemma 8.4. *The set* $\Psi^R(\mathbb{L})$ *of all pairs* (α, A)*, where* $\alpha \in \mathrm{PsAut}^R(\mathbb{L})$ *and* $A \in \mathbb{L}$ *is its companion, is a group with identity element* $(\mathrm{id}, 1)$ *and the following group operations:*

$$\text{product:} \quad (\alpha_1, A_1)(\alpha_2, A_2) = (\alpha_1 \circ \alpha_2, \alpha_1(A_2) A_1) \tag{8.46a}$$

$$\text{inverse:} \quad (\alpha, A)^{-1} = \left(\alpha^{-1}, \alpha^{-1}\left(A^{\lambda}\right)\right) = \left(\alpha^{-1}, \left(\alpha^{-1}(A)\right)^{\rho}\right) \tag{8.46b}$$

Proof. Indeed, it is easy to see that $\alpha_1(A_2) A_1$ is a companion of $\alpha_1 \circ \alpha_2$, that (8.46a) is associative, and that $(\mathrm{id}, 1)$ is the identity element with respect to it. Also, it is easy to see that

$$(\alpha, A)\left(\alpha^{-1}, \alpha^{-1}\left(A^{\lambda}\right)\right) = (\mathrm{id}, 1).$$

On the other hand, setting $B = \alpha^{-1}\left(A^{\lambda}\right)$, we have

$$\begin{aligned} B &= \alpha^{-1}(1) B = \alpha^{-1}\left(A^{\lambda} A\right) B \\ &= \alpha^{-1}\left(A^{\lambda}\right) \cdot \alpha^{-1}(A) B \\ &= B \cdot \alpha^{-1}(A) B. \end{aligned}$$

Cancelling A on both sides on the left, we see that $B = \left(\alpha^{-1}(A)\right)^{\rho}$.

Let $C^R(\mathbb{L})$ be the set of elements of $\mathbb{L}$ that are a companion for a right pseudoautomorphism. Then, (8.46a) shows that there is a left action of $\Psi^R(\mathbb{L})$ on $C^R(\mathbb{L})$ given by:

$$\Psi^R(\mathbb{L}) \times C^R(\mathbb{L}) \longrightarrow C^R(\mathbb{L}) \tag{8.47a}$$

$$((\alpha, A), B) \mapsto (\alpha, A) B = \alpha(B) A. \tag{8.47b}$$

This action is transitive, because if $A, B \in C^R(\mathbb{L})$, then exist $\alpha, \beta \in \mathrm{PsAut}^R(\mathbb{L})$, such that $(\alpha, A), (\beta, B) \in \Psi^R(\mathbb{L})$, and hence $\left((\beta, B)(\alpha, A)^{-1}\right) A = B$. Similarly, $\Psi^R(\mathbb{L})$ also acts on all of $\mathbb{L}$. Let $h = (\alpha, A) \in \Psi^R(\mathbb{L})$, then for any $p \in \mathbb{L}$, $h(p) = \alpha(p) A$. This is in general non-transitive, but a faithful action (assuming $\mathbb{L}$ is non-trivial). Using this, the definition of (8.43) can be rewritten as

$$h(pq) = \alpha(p) h(q) \tag{8.48}$$

and hence the quotient relations (8.44) may be rewritten as

$$h(q \backslash p) = \alpha(q) \backslash h(p) \tag{8.49a}$$

$$\alpha(p/q) = h(p)/h(q). \tag{8.49b}$$

If $\Psi^R(\mathbb{L})$ acts transitively on $\mathbb{L}$, then $C^R(\mathbb{L}) \cong \mathbb{L}$, since every element of $\mathbb{L}$ will be a companion for some right pseudoautomorphism. In that case, $\mathbb{L}$ is known as a (*right*) *G-loop*. Note that usually a loop is known as a G-loop is every element of $\mathbb{L}$ is a companion for a right pseudoautomorphism and for a left pseudoautomorphism (36). However, in this paper we will only be concerned with right pseudoautomorphisms, so for brevity we will say $\mathbb{L}$ is a G-loop if $\Psi^R(\mathbb{L})$ acts transitively on it.

There is another action of $\Psi^R(\mathbb{L})$ on $\mathbb{L}$ - which is the action by the pseudoautomorphism. This is a non-faithful action of $\Psi^R(\mathbb{L})$, but corresponds to a faithful action of $\mathrm{PsAut}^R(\mathbb{L})$. Namely, let $h = (\alpha, A) \in \Psi^R(\mathbb{L})$, then h acts on $p \in \mathbb{L}$ by $p \mapsto \alpha(p)$. To distinguish these two actions, we make the following definitions.

Definition 8.5. *Let* $\mathbb{L}$ *be a loop and let* $\Psi^R(\mathbb{L})$ *the group of right pseudoautomorphism pairs.* $\mathbb{L}$ *admits two left actions of* $\Psi^R(\mathbb{L})$ *on itself. Let* $h = (\alpha, A) \in \Psi^R(\mathbb{L})$ *and* $p \in \mathbb{L}$.

1. *The* full *action is given by* $(h, p) \mapsto h(p) = \alpha(p) A$. *The set* $\mathbb{L}$ *together with this action of* $^R(\mathbb{L})$ *will be denoted by* $\mathring{\mathbb{L}}$.
2. *The* partial *action, given by* $(h, p) \mapsto h'(p) = \alpha(p)$. *The set* $\mathbb{L}$ *together with this action of* $^R(\mathbb{L})$ *will be denoted by* $\mathbb{L}$ *again.*

Remark 8.6. *From (8.48), these definitions suggest that the loop product on* $\mathbb{L}$ *can be regarded as a map* $\cdot : \mathbb{L} \times \mathring{\mathbb{L}} \longrightarrow \mathring{\mathbb{L}}$. *This bears some similarity to Clifford product structure on spinors, however without the linear structure, but instead with the constraint that* $\mathbb{L}$ *and* $\mathring{\mathbb{L}}$ *are identical as sets. This however allows to define left and right division. '*

Now let us consider several relationships between the different groups associated to $\mathbb{L}$. First of all define the following maps:

$$\begin{aligned} \iota_1 &: \mathrm{Aut}(\mathbb{L}) \hookrightarrow \Psi^R(\mathbb{L}) \\ \gamma &\mapsto (\gamma, 1) \end{aligned} \tag{8.50}$$

and

$$\begin{aligned} \iota_2 &: \mathcal{N}^R(\mathbb{L}) \hookrightarrow \Psi^R(\mathbb{L}) \\ C &\mapsto (\mathrm{id}, C), \end{aligned} \tag{8.51}$$

The map ι_1 is clearly injective and is a group homomorphism, so $\iota_1(\mathrm{Aut}(\mathbb{L}))$ is a subgroup of $\Psi^R(\mathbb{L})$. On the other hand, if $A, B \in \mathcal{N}^R(\mathbb{L})$, then in $\Psi^R(\mathbb{L})$, $(\mathrm{id}, A)(\mathrm{id}, B) = (\mathrm{id}, BA)$, so ι_2 is an antihomomorphism from $\mathcal{N}^R(\mathbb{L})$ to $\Psi^R(\mathbb{L})$ and thus a homomorphism from the opposite group $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$. So, $\iota_2\left(\mathcal{N}^R(\mathbb{L})\right)$ is a subgroup of $\Psi^R(\mathbb{L})$ that is isomorphic to $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$.

Using (8.50) let us define a right action of $\mathrm{Aut}(\mathbb{L})$ on $\Psi^R(\mathbb{L})$. Given $\gamma \in \mathrm{Aut}(\mathbb{L})$ and $(\alpha, A) \in \Psi^R(\mathbb{L})$, we define

$$(\alpha, A) \cdot \gamma = (\alpha, A)\iota_1(\gamma) = (\alpha \circ \gamma, A). \tag{8.52}$$

Similarly, (8.51) allows to define a left action of $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$,and hence a right action of $\mathcal{N}^R(\mathbb{L})$, on $\Psi^R(\mathbb{L})$:

$$C \cdot (\alpha, A) = \iota_2(C)(\alpha, A) = (\alpha, AC). \tag{8.53}$$

The actions (8.52) and (8.53) commute, so we can combine them to define a left action of $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$. Indeed, given $\gamma \in \mathrm{Aut}(\mathbb{L})$ and $C \in \mathcal{N}^R(\mathbb{L})$,

$$(\alpha, A) \cdot (\gamma, C) = \iota_2(C)(\alpha, A)\iota_1(\gamma) = (\alpha \circ \gamma, AC). \tag{8.54}$$

Remark 8.7. *Since any element of* $\mathcal{N}^R(\mathbb{L})$ *is a right companion for any automorphism, we can also define the semi-direct product subgroup*

$\iota_1(\mathrm{Aut}(\mathbb{L})) \ltimes \iota_2(\mathcal{N}^R(\mathbb{L})) \subset \Psi^R(\mathbb{L})$. *Suppose* $\beta, \gamma \in \mathrm{Aut}(\mathbb{L})$ *and* $B, C \in \mathcal{N}^R(\mathbb{L})$, *then in this semi-direct product,*

$$(\beta, B)(\gamma, C) = (\beta\circ\gamma, \beta(C)B).$$

Lemma 8.8. *Given the actions of* $\mathrm{Aut}(\mathbb{L})$ *and* $\mathcal{N}^R(\mathbb{L})$ *on* $\Psi^R(\mathbb{L})$ *as in (8.52) and (8.53), respectively, we have the following properties.*

1. $\Psi^R(\mathbb{L})/\mathrm{Aut}(\mathbb{L}) \cong C^R(\mathbb{L})$ *as* $\Psi^R(\mathbb{L})$*-sets.*
2. *The image* $\iota_2(\mathcal{N}^R(\mathbb{L}))$ *is a normal subgroup of* $\Psi^R(\mathbb{L})$ *and hence*

$$\Psi^R(\mathbb{L})/\mathcal{N}^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L}).$$

3. *Moreover,*

$$\Psi^R(\mathbb{L})/\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L})/\mathrm{Aut}(\mathbb{L}) \cong C^R(\mathbb{L})/\mathcal{N}^R(\mathbb{L})$$

where equivalence is as $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$*-sets.*

Proof. Suppose $\mathbb{L}$ is a loop.

1. Consider the projection on the second component $\mathrm{prj}_2 : \Psi^R(\mathbb{L}) \longrightarrow C^R(\mathbb{L})$ under which $(\alpha, A) \mapsto A$. Both $\Psi^R(\mathbb{L})$ and $C^R(\mathbb{L})$ are left $\Psi^R(\mathbb{L})$-sets, since both admit a left $\Psi^R(\mathbb{L})$ action - $\Psi^R(\mathbb{L})$ acts on itself by left multiplication and acts on $C^R(\mathbb{L})$ via the action (8.47). Hence, prj_2 is a $\Psi^R(\mathbb{L})$-equivariant map (i.e. a G-set homomorphism). On the other hand, given the action (8.52) of $\mathrm{Aut}(\mathbb{L})$ on $\Psi^R(\mathbb{L})$, we easily see that two pseudoautomorphisms have the same companion if and only if they lie in the same orbit of $\mathrm{Aut}(\mathbb{L})$. Thus, prj_2 descends to a $\Psi^R(\mathbb{L})$-equivariant bijection $\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \longrightarrow C^R(\mathbb{L})$, so that $\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \cong C^R(\mathbb{L})$ as $\Psi^R(\mathbb{L})$-sets.
2. It is clear that $C \in C^R(\mathbb{L})$ is a right companion of the identity map id if and only if $C \in \mathcal{N}^R(\mathbb{L})$. Now, let $\nu = (\mathrm{id}, C) \in \iota_2(\mathcal{N}^R(\mathbb{L}))$ and $g = (\alpha, A) \in \Psi^R(\mathbb{L})$. Then,

$$g\nu g^{-1} = (\alpha, A)(\mathrm{id}, C)\left(\alpha^{-1}, \alpha^{-1}\left(A^{\lambda}\right)\right) = \left(\mathrm{id}, A^{\lambda} \cdot \alpha(C)A\right). \quad (8.55)$$

In particular, this shows that $g\nu g^{-1} \in \iota_2(\mathcal{N}^R(\mathbb{L}))$ since $A^{\lambda} \cdot \alpha(C)A$ is the right companion of id. Thus indeed, $\iota_2(\mathcal{N}^R(\mathbb{L}))$ is a normal subgroup of $\Psi^R(\mathbb{L})$. Now consider the projection on the first component $\mathrm{prj}_1 : \Psi^R(\mathbb{L}) \longrightarrow \mathrm{PsAut}^R(\mathbb{L})$ under which $(\alpha, A) \mapsto \alpha$. This is clearly a group homomorphism with kernel $\iota_2(\mathcal{N}^R(\mathbb{L}))$. Thus, ${}^R(\mathbb{L})^{\mathrm{op}} \backslash \Psi^R(\mathbb{L}) \cong \Psi^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L})$.
3. Since the actions of $\mathcal{N}^R(\mathbb{L})$ and $\mathrm{Aut}(\mathbb{L})$ on $\Psi^R(\mathbb{L})$ commute, the action of $\mathrm{Aut}(\mathbb{L})$ descends to $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}} \backslash \Psi^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L})$ and the action of $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$ descends to $\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \cong C^R(\mathbb{L})$. Since the left action of $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$ on $\Psi^R(\mathbb{L})$ corresponds to an action by right multiplication on $C^R(\mathbb{L})$, we find that there is a bijection $\mathrm{PsAut}^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \longrightarrow C^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L})$.

   Suppose $(\alpha, A) \in \Psi^R(\mathbb{L})$ and let $[\alpha]_{\mathrm{Aut}(\mathbb{L})} \in \mathrm{PsAut}^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L})$ be the orbit of α under the action of $\mathrm{Aut}(\mathbb{L})$ and let $[A]_{\mathcal{N}^R(\mathbb{L})} \in C^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L})$ be the orbit of A under the action of $\mathcal{N}^R(\mathbb{L})$. Then the bijection is given by $[\alpha]_{\mathrm{Aut}(\mathbb{L})} \mapsto [A]_{\mathcal{N}^R(\mathbb{L})}$. Moreover, each of these orbits also corresponds to the orbit of (α, A) under the right action of $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$ on $\Psi^R(\mathbb{L})$. These quotients preserve actions of $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$ on corresponding sets and thus these coset spaces are equivalent as $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$-sets.

The above relationships between the different groups are summarized in Figure 2.

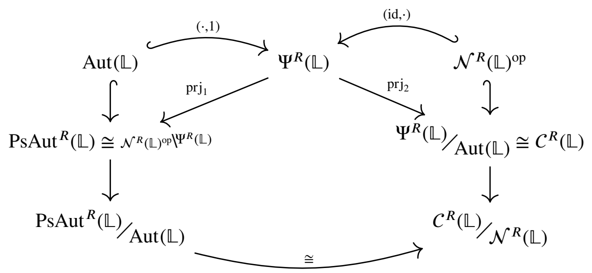


Fig. 2. Groups related to the loop $\mathbb{L}$

Conclusion

In this paper, we try to combine partial differential equations with algebraic geometry and other content through YH-DIE (also see (48)). In order to achieve this ambition, we tried to study from the perspectives of Algebraic Geometry, Differential Geometry and Analysis of PDEs. YH-DIE is still a relatively young research object in mathematics, and we hope that more mathematician will participate and develop it into a mature branch.

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