Sets, Classes and Functions

This chapter includes Section 0.1 to 0.3 of the original book.

Definition 1.0.1. A *set* A is a class such that there exists a class B with $A \in B$. A class that is not a set is a *proper class*.

From the definition above we can find that sets and classes are not the same. In fact, the distinction between sets and proper classes is not too clear.

Example 1.0.2 (by B. Russel)
Consider the class $M = \{X \mid X \text{ is a set and } X \notin X\}$. The statement $X \notin X$ is not unreasonable since many sets satisfy it (for example, the set of all books is not a book). M is a proper class. For if M were a set, then either $M \in M$ or $M \notin M$. But by the definition of M, $M \in M$ implies $M \notin M$ and $M \notin M$ implies $M \in M$. Thus in either case the assumption that M is a set leads to an untenable paradox: $M \in M$ and $M \notin M$.

Definition 1.0.3. The *power axiom* asserts that for every set A the class $P(A)$ of all subsets of A is itself a set. $P(A)$ is called the *power set* of A; it is also denoted 2^A.

Definition 1.0.4. A *family* of sets indexed by (the nonempty class) I is a collection of sets A_i, one for each $i \in I$ (denoted $\{A_i \mid i \in I\}$).Given such a family, its *union* and *intersection* are defined to be respectively the classes

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\},$$

$$\bigcap_{i\in I} A_i = \{x \mid A_i \text{ for all } i \in I\}$$

Note 1 — It's worth mentioning, indexing class I may not be finite and not even be countable. Also, 'suitable axioms' insure that $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ are actually sets.

Definition 1.0.5. If A and B are classes, the *relative complement* of A in B is the following subclass of B:

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

If all the classes under discussion are subsets of some fixed set U (called the universe of discussion), then $U - A$ is denoted A' and called simply the *complement* of A.

Theorem 1.0.6 (Properties of sets)

$$A \bigcap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I} (A \bigcap B_i) \text{ and } A \bigcup (\bigcap_{i\in I} B_i) = \bigcap_{i\in I} (A \bigcup B_i)$$

$$(\bigcup_{i\in I} A_i)' = \bigcap_{i\in I} A_i' \text{ and } (\bigcap_{i\in I} A_i)' = \bigcup_{i\in I} A_i' \text{ (DeMorgan's Laws)}$$

$$A \bigcup B = B \iff A \subset B \iff A \bigcap B = A$$

Proof. Easy to prove. □

Definition 1.0.7. Let $f : A \to B$. If $S \subset A$, the image of S under f (denoted $f(S)$) is

$$f(S) = \{b \in B \mid b = f(a) \text{ for some } a \in S\}$$

If $T \subset B$, the *inverse image* of T under f (denoted $f^{-1}(T)$) is

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}$$

Example 1.0.8

If $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$, then $f(\mathbb{R}) = [0, \infty)$ and $f^{-1}(\{9\}) = \{3, 3\}$

A function $f : A \to B$ is said to be *injective* (or one-to-one) provided for all $a, a' \in A$, $a \neq a' \Rightarrow f(a) \neq f(a')$. A function is *surjective* (or onto) provided $f(A) = B$. A function f is said to be *bijective* (or a bijection or a one-to-one correspondence) if it is both injective and surjective.

Proposition 1.0.9

$$f \text{ and } g \text{ injective} \Rightarrow gf \text{ is injective;} \tag{1.1}$$
$$f \text{ and } g \text{ surjective} \Rightarrow gf \text{ is surjective;} \tag{1.2}$$
$$gf \text{ is injective} \Rightarrow f \text{ is injective;} \tag{1.3}$$
$$gf \text{ is surjective} \Rightarrow g \text{ is surjective.} \tag{1.4}$$

Theorem 1.0.10

Let $f : A \to B$ be a function, with A nonempty.

(i) f is injective if and only if there is a map $g : B \to A$ such that $gf = 1_A$[a].

(ii) If A is a set, then f is surjective if and only if there is a map $h : B \to A$ such that $fh = 1_B$.

[a] identity function ($1_A : A \to A, a \mapsto a$)

Proof. ($\Leftarrow$)Each identity function is bijective, so from 1.3 and 1.4 can easily prove.
($\Rightarrow$)First let's prove (i). Since f is injective, for each $b \in f(A)$ there exists a unique $a \in A$

such that $f(a) = b$. Choose a fixed $a_0 \in A$ (no matter what a_0 is), then define $g : B \to A$ by

$$g(b) = \begin{cases} a & \text{if } b \in f(A) \text{ and } f(a) = b \\ a_0 & \text{if } b \notin f(A) \end{cases}$$

Then the map g meets the conditions. Next let's prove (ii). Since g is surjective, the set $f^{-1}(b) \subset A$ is nonempty for each $b \in B$. For each $b \in B$ choose $a_b \in f^{-1}(b)$[1]. The map $h : B \to A, b \mapsto a_b$ meets the conditions. □

By 1.0.10 if A is a set and $f : A \to B$ a function, then f is bijective $\iff$ f has a two-sided inverse.

[1]This requires the Axiom of Choice. See 3.0.1