Informal Notes on Mathematics
2022.03.19

A Rewiew at Calculus

Having defined real numbers(may mention in the following notes), the first thing we will study is sequences. We will want to study what it means for a sequence to converge. Intuitively, we would like to say that $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ converges to 0, while $1, 2, 3, 4, \cdots$ diverges. However, the actual formal definition of convergence is rather hard to get right, and historically there have been failed attempts that produced spurious results.

Definition Sequence

A *sequence* is, formally, a function $a : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$). Usually (i.e. always), we write a_n instead of $a(n)$. Instead of a, we usually write it as (a_n), $(a_n)_1^\infty$ or $(a_n)_{n=1}^\infty$ to indicate it is a sequence.

Definition Limits of seuqence

Let (a_n) be a sequence and $\ell \in \mathbb{R}$. Then a_n *converges to* ℓ, *tends to* ℓ, or $a_n \to \ell$, if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that whenever $n > N$, we have $|a_n - \ell| < \varepsilon$. In symbols, this says

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)\ |a_n - \ell| < \varepsilon.$$

We say ℓ is the *limit* of (a_n).

Let's take a look at an example from Demidovich. Suppose $x_n = \dfrac{n}{n+1} (n \in \mathbb{N})$, prove that$\lim_{x \to \infty} x_n = 1$

Proof: That is, proof that that for any given $\varepsilon > 0$, find out the $N = N(\varepsilon)$ that when $n > N$,we have $|x_n - 1| < \varepsilon$

Since $|x_n - 1| = \dfrac{1}{n+1}$, we only need to let $\dfrac{1}{n+1} < \varepsilon$, i.e. $n > \dfrac{1}{\varepsilon} - 1$. So let $N = N(\varepsilon) = \left[\dfrac{1}{\varepsilon}\right]$. □

Definition Limit of function

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say

$$\lim_{x \to a} f(x) = \ell,$$

or $f(x) \to \ell$ as $x \to a$, if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)\ 0 < |x - a| < \delta \Rightarrow |f(x) - \ell| < \varepsilon. \quad (1)$$

We couldn't care less what happens when $x = a$, hence the strict inequality $0 < |x - a|$. In fact, f doesn't even have to be defined at $x = a$.

Definition Derivative of function

The *derivative* of a function $f(x)$ with respect to x, interpreted as the rate of change of $f(x)$ with x, is

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

A function $f(x)$ is differentiable at x if the limit exists (i.e. the left-hand and right-hand limits are equal).

Definition O and o notations

(i) "$f(x) = o(g(x))$ as $x \to x_0$" if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$. Intuitively, $f(x)$ is much smaller than $g(x)$.

(ii) "$f(x) = O(g(x))$ as $x \to x_0$" if $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$. Intuitively, $f(x)$ is about as big as $g(x)$.

Note that for $f(x) = O(g(x))$ to be true, $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ need not exist.

Usually, x_0 is either 0 or infinity. Clearly, we have $f(x) = o(g(x))$ implies $f(x) = O(g(x))$.

Theorem 1 *(Chain rule)*
Given $f(x) = F(g(x))$, then

$$\frac{df}{dx} = \frac{dF}{dg}\frac{dg}{dx}.$$

Theorem 2 *(Product rule)*
Give $f(x) = u(x)v(x)$. Then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Theorem 3 *(Quotient rule)*
Give $f(x) = \frac{u(x)}{v(x)}$, then

$$f'(x) = \frac{u'v - uv'}{v^2}$$

Theorem 4 *(Leibniz's rule)*
Given $f = uv$, then

$$f^{(n)}(x) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)},$$

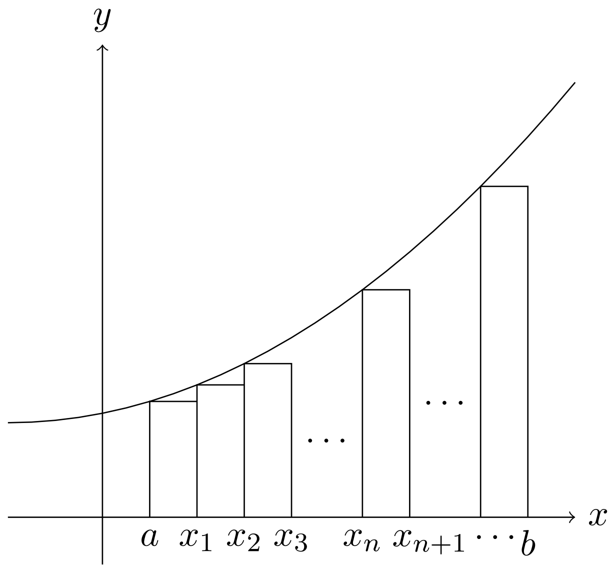
where $f^{(n)}$ is the n-th derivative of f.

Definition Integration

An *integral* is the limit of a sum, e.g.

$$\int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum_{n=0}^{N} f(x_n) \Delta x.$$

For example, we can take $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$. Note that an integral need not be defined with this particular Δx and x_n. The term ""integral" simply refers to any limit of a sum (The usual integrals we use are a special kind known as Riemann integral, which we will study formally in analysis). Pictorially, we have



The area under the graph from x_n to x_{n+1} is $f(x_n)\Delta x + O(\Delta x^2)$. Provided that f is differentiable, the total area under the graph from a to b is

$$\lim_{N \to \infty} \sum_{n=0}^{N-1} (f(x_n)\Delta x) + N \cdot O(\Delta x^2) = \lim_{N \to \infty} \sum_{n=0}^{N-1} (f(x_n)\Delta x) + O(\Delta x) = \int_a^b f(x)\ dx$$

Theorem 5 *(Fundamental Theorem of Calculus)*

Let $F(x) = \int_a^x f(t)\ dt$. Then $F'(x) = f(x)$

Proof:

$$
\begin{aligned}
\frac{d}{dx}F(x) &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\ dt - \int_a^{x} f(t)\ dt\right] \\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\ dt \\
&= \lim_{h\to 0}\frac{1}{h}[f(x)h + O(h^2)] \\
&= f(x)
\end{aligned}
$$

OK. That's all for today.