Informal Notes on Mathematics
2022.03.21

Some (Pre-) Complex Analysis

Addition: parallelogram law

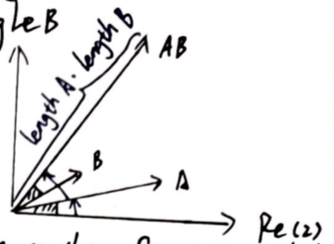
$x = a + bi$, $y = c + di$, $x + y = (a+c) + (b+d)i$

Multiplication: $A = x + iy$, $B = a + ib$

$$AB = (x+iy)(a+ib) = xa + i(ya+xb) + i^2(yb)$$
$$= (xa - yb) + (ya + xb)i$$

The length of AB = length A · length B

The angle of AB = angle A + angle B
(angle A: arg A)



If we write the complex number in another form, it will be obvious.

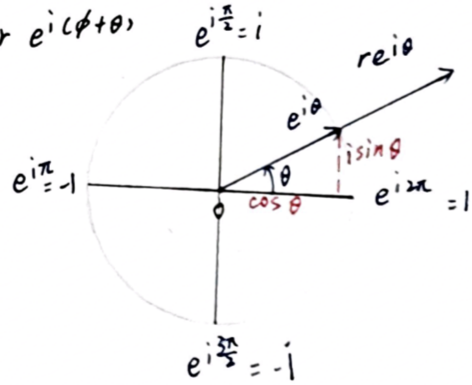
Just like polar coordinates, let $r = |z| = \sqrt{x^2+y^2}$, $\theta = \arg z$

$A = R\angle\phi$, $B = r\angle\theta$, $AB = (Rr)\angle(\phi+\theta)$

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

We can now write $z = r\angle\theta$ as $z = re^{i\theta}$

So we have $AB = (Rr)\, e^{i\theta} \cdot e^{i\phi} = Rr\, e^{i(\phi+\theta)}$



But how we get the formula?

Using Talor's series is an easy way

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\Downarrow$$
$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$
$$= \cos\theta + i\sin\theta$$

We can then have $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$

However, there is a better way (by Needham)

First of all, we know that

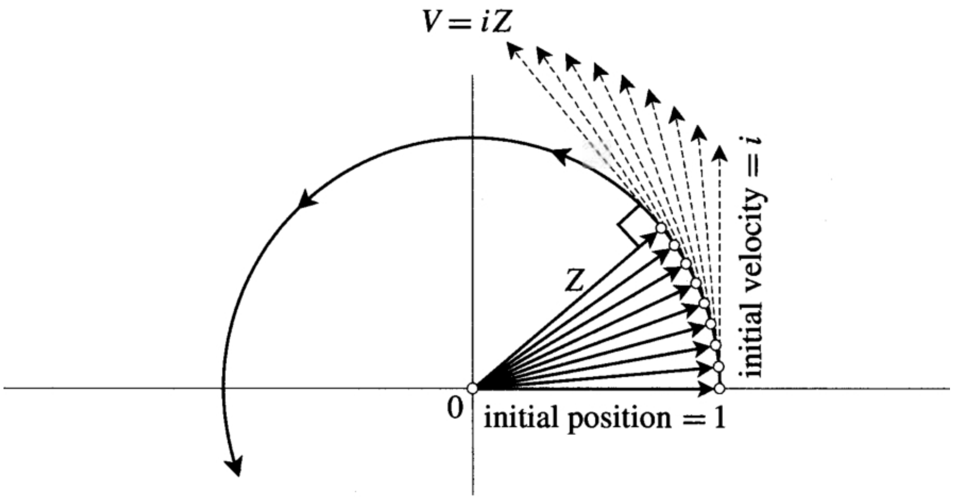
$$\frac{d}{dt}\ e^{it} = ie^{t}$$

Herr we consider t as time. Recall from Physics, velocity $V(t)$ is the vector—now thought of as a complex number—whose length and direction are given by the instantaneous speed, and the instantaneous direction of motion (tangent to the trajectory), of the moving particle. The figure shows the movement M of the particle between time t and $t+\delta$,and this should make it clear that

$$\frac{d}{dt}Z(t) = \lim_{\delta\to 0}\frac{Z(t+\delta)-Z(t)}{\delta} = \lim_{\delta\to 0}\frac{M}{\delta} = V(t)$$

Thus, given a complex function Z(t) of a real variable t,we can always visualize Z as the position of a moving particle,and $\frac{dZ}{dt}$ its velocity.

Now $Z(t) = e^{it}$, so we have velocity= V= iZ =position, rotated through a right angle. Since the initial position of the particle is $Z(0) = e^{0} = 1$,its initial velocity is i, and so it is moving vertically upwards. A split second later the particle will have moved very slightly in this direction, and its new velocity will be at right angles to its new position vector. Continuing to construct the motion in this way, it is clear that the particle will travel round the unit circle.



And here it is!

If $C_1C_2C_3 \cdots C_n$ is a regular n-gon inscribed in a circle of unit radius centred at O, and P is the point on OC_1 at distance x from O, then $U_n(x) = PC_1 \cdot PC_2 \cdots PC_n$.

This is Cotes' result. Unfortunately, he stated it without proof, and he left no clue as to how he discovered it. Thus we can only speculate that he may have been guided by an argument like the one we have just supplied[9].

Since the vertices of the regular n-gon will always come in symmetric pairs that are equidistant from P, the examples in [18] make it clear that Cotes' result is indeed equivalent to factorizing $U_n(x)$ into real linear and quadratic factors.

Recovering from our feigned bout of amnesia concerning complex numbers and their geometric interpretation, Cotes' result becomes simple to understand and to prove. Taking O to be the origin of the complex plane, and C_1 to be 1, the vertices of Cotes' n-gon are given by $C_{k+1} = e^{ik(2\pi/n)}$. See [19], which illustrates the case $n = 12$. Since $(C_{k+1})^n = e^{ik2\pi} = 1$, all is suddenly clear: *The vertices of the regular n-gon are the n complex roots of $U_n(z) = z^n - 1$.* Because the solutions of $z^n - 1 = 0$ may be written formally as $z = \sqrt[n]{1}$, the vertices of the n-gon are called the n^{th} *roots of unity*.

By Descartes' Factor Theorem, the complete factorization of $(z^n - 1)$ is therefore

$$z^n - 1 = U_n(z) = (z - C_1)(z - C_2) \cdots (z - C_n),$$

with each conjugate pair of roots yielding a real quadratic factor,

$$\left(z - e^{ik(2\pi/n)}\right)\left(z - e^{-ik(2\pi/n)}\right) = z^2 - 2z\cos\left[\frac{2k\pi}{n}\right] + 1.$$

Each factor $(z - C_k) = R_k\, e^{i\phi_k}$ may be viewed (cf. [17a]) as a complex number connecting a vertex of the n-gon to z. Thus, if P is an arbitrary point in the plane

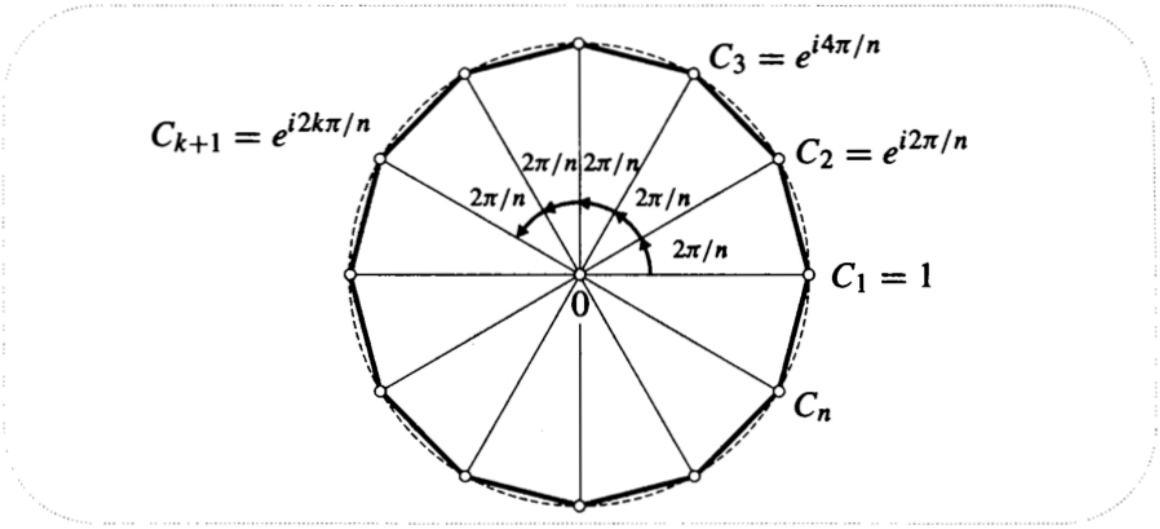


Figure [19]

(not merely a point on the real axis), then we obtain the following generalized form of Cotes' result:

$$U_n(P) = [PC_1 \cdot PC_2 \cdots PC_n]\, e^{i\Phi},$$

where $\Phi = (\phi_1 + \phi_2 + \cdots + \phi_n)$. If P happens to be a real number (again supposed greater than 1) then $\Phi = 0$ [make sure you see this], and we recover Cotes' result.

And what I want to say is, similar to Cote's result , I got another result.

Theorem 1

Polynomial $x^{2n} + x^n + 1$ *has a factor* $x^2 + x + 1$ *only when* $n \bmod 3 \not\equiv 0$.

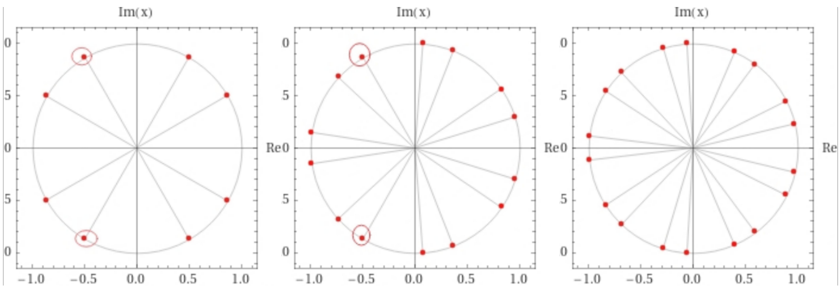


Figure 1: $n = 4, n = 7$ and $n = 9$

This is also through the analysis of the angle of the roots in the complex plane.