Informal Notes on Mathematics
2022.03.23

Some Complex Analysis as well

In the ten analysis, we can always encounter some functions that make us very uncomfortable. For example:
(a) The *Devil's Staircase* (or *Cantor function*) is a continuous function $H:[0,1]\to[0,1]$ which has derivative zero "almost everywhere", yet $H(0)=0$ and $H(1)=1$.
(b) The *Weierstraβ function*

$$x\mapsto\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}\cos(114514^{n}\pi x)$$

is continuous everywhere but differentiable nowhere.
(c) The function

$$x\mapsto\begin{cases}x^{100} & x\geq 0\\ -x^{100} & x<0\end{cases}$$

has the first 99 derivatives but not the 100th one.
(d) If a function has all derivatives (we call these *smooth functions*), then it has a Taylor series. But for real functions that Taylor series might still be wrong. The function

$$x\mapsto\begin{cases}e^{-1/x} & x>0\\ 0 & x\leq 0\end{cases}$$

has derivatives at every point. But if you expand the Taylor series at $x=0$, you get $0+0x+0x^2+\ldots$, which is wrong for any $x>0$ (even $x=0.0001$).

We restrict our attention to differentiable functions called *holomorphic functions*. It turns out that the multiplication on $\mathbb{C}$ makes all the difference. Opposite the real functions, knowing tiny amounts of data about the function can determine all its values.

Definition 1
Let $f:U\to\mathbb{C}$ be a complex function. Then for some $z_0\in U$, we define the ***derivative*** *at z_0 to be*

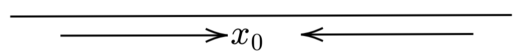
$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}.$$

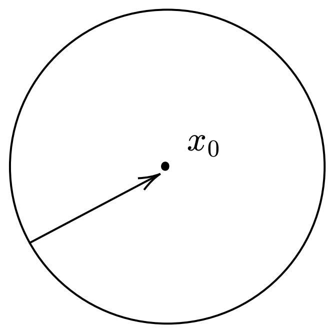
Note that this limit may not exist; when it does we say f is ***differentiable*** *at z_0.*

好我们来看一些需要注意的地方。从定义上看这和我们在实分析里学的微分定义一模一样，因此我们可以猜想这个“复”极限就是指对于每个$\varepsilon>0$,有一个$\delta>0$使得

$$0 < |h| < \delta \Longrightarrow \left| \frac{f(z_0 + h) - f(z_0)}{h} - L \right| < \varepsilon$$

但是注意到这时候这个δ邻域已经不是直线上的一段区间了，而是一个圆。这部分可参见点集拓扑。在处理实极限时，我们要让左极限和右极限同时趋于x_0，可以理解成从直线两端向中间某点趋近，但是复平面内不同；它要求从各个方向。这里简单画一个示意图



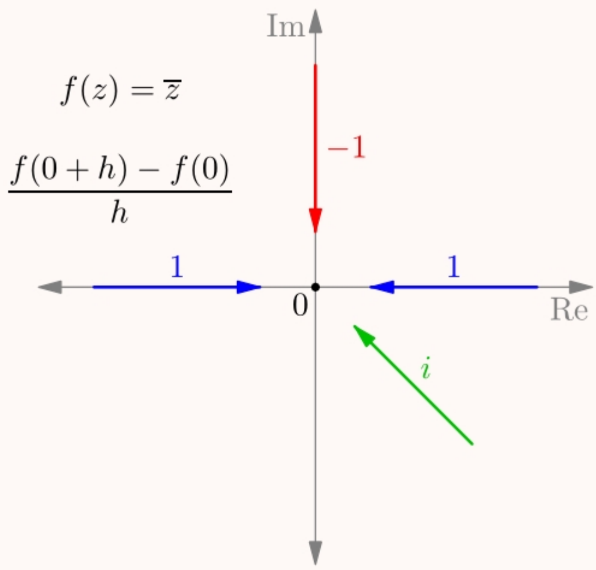


Example 31.2.1 (Important: conjugation is *not* holomorphic)

Let $f(z) = \overline{z}$ be complex conjugation, $f : \mathbb{C} \to \mathbb{C}$. This function, despite its simple nature, is not holomorphic! Indeed, at $z = 0$ we have,

$$\frac{f(h) - f(0)}{h} = \frac{\overline{h}}{h}.$$

This does not have a limit as $h \to 0$, because depending on "which direction" we approach zero from we have different values.



Definition 2

If a function $f : U \to \mathbb{C}$ *is complex differentiable at all the points in its domain it is called holomorphic. In the special case of a holomorphic function with domain* $U = \mathbb{C}$,*we call the function entire.*

In all the examples below, the derivative of the function is the same as in their real analogues (e.g. the derivative of e^z is e^z).

(a) Any polynomial $z \mapsto z^n + c_{n-1}z^{n-1} + \cdots + c_0$ is holomorphic.

(b) The complex exponential $\exp : x + yi \mapsto e^x(\cos y + i \sin y)$ can be shown to be holomorphic.

(c) $\sin$ and $\cos$ are holomorphic when extended to the complex plane by $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

(d) As usual, the sum, product, chain rules and so on apply, and hence sums, products, nonzero quotients, and compositions of holomorphic functions are also holomorphic.