Informal Notes on Mathematics

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写一些奥赛入门的东西吧。积分什么的拖了一周倒底也没写成笔记，可能我注定是积佬的反面（^op）吧。今天本来看了一些 affine 几何。像 affine space，direction，parallel，affinely independent 什么的定义总让我想起抽代，点拓还有向量空间这类非常 based upon set theory 的东西，倒是更若我喜欢。但没办法，目前最近的目标还是丘班，还是得去练烦人的基础和计算。Шиэнь та ма дэ фанжэнь.

1、证明的一些注意事项

(i) "find all" 要证两部分

1. 证明以下几种结果成立…… (find)
2. 证明以上结果是"所有"结果 (only ones)…… (all)

Example. 一个看起来很傻的问题

Find all real numbers x such that $3x+2=17$.

解：($\Rightarrow$) $3x+2=17 \Rightarrow 3x=15 \Rightarrow x=5$ Step 1.

($\Leftarrow$) $x=5 \Rightarrow 3x=15 \Rightarrow 3x+2=17$ Step 2.

从中我们可以看出在此类情况下"find all"相当于证明"当且仅当 (iff)"

(所以直接用 $\Leftrightarrow$ 就完事了)

另一类情况就是排掉多余的解，好比我们检验增根。

Example. USAJMO 2011

Find all positive integers n such that $2^n+12^n+2011^n$ is a perfect square.

解：观察得 $n=1$ 时成立。 Step 1.

如果 $n>1$ 且为奇，则 $2^n+12^n+2011^n \equiv 3 \pmod 4$，不是完全平方

如果 $n>1$ 且为偶，则 $2^n+12^n+2011^n \equiv 2 \pmod 3$，不是完全平方 (Step 2.)

$\therefore$ 仅当 $n=1$ 时成立

(ii) 最优问题要证两部分

1. 证明一个边界（上界/下界）
2. 证明上述边界能取到（最大值/最小值）

这就是不等式问题恶心的一个地方。等以后遇到再说吧。(USAMO 2010)

(iii) "存在唯一"要证两部分

显然，证"存在"和"唯一"。而实际上也是证 iff，唯一 ($\Rightarrow$)，存在 ($\Leftarrow$)

(iv) 最小的集合

这个似乎在点拓、抽代中更常用些。即证集合 S 被其它所有满足条件的集合包含。

(参见 Putnam 2017 那道集合论。然后如何证"最小"暂时还是让我比较迷惑，这道题姑且日后再谈。)

2、基本不等式

cyclic sum notation: $\sum_{cyc}$　　symmetric sum notation $\sum_{sym}$

Example: (three variable)

$$\sum_{cyc} a^2 = a^2+b^2+c^2$$

$$\sum_{cyc} a^2b = a^2b+b^2c+c^2a$$

$$\sum_{sym} a^2 = a^2+a^2+b^2+b^2+c^2+c^2$$

$$\sum_{sym} a^2b = a^2b+a^2c+b^2a+b^2c+c^2a+c^2b$$

AM-GM: ($a_1, a_2 \ldots a_n$ 非负)

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1\cdots a_n}$$

Problem: Canadian Olympiad 2002

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \geq a+b+c$$

解: 根据 AM-GM，我们有

$$\frac{2\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}}{4} \geq \sqrt[4]{\frac{a^3\cdot a^3\cdot b^3\cdot c^3}{bc\cdot bc\cdot ca\cdot ab}} = a$$

同理对 b、c 操作，将不等式相加，

得 $\frac{(2+1+1)}{4}\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right) \geq a+b+c$

即原命题得证。

Muirhead's inequality: $a_1, a_2 \ldots a_n$ 非负且序列 $\{x_n\} \succ \{y_n\}$，我们有

$$\sum_{sym} a_1^{x_1}a_2^{x_2}\cdots a_n^{x_n} \geq \sum_{sym} a_1^{y_1}a_2^{y_2}\cdots a_n^{y_n}$$

(此处说明下"$\succ$"是什么意思：

$x_1 \geq x_2 \geq \cdots \geq x_n$，　$y_1 \geq y_2 \geq \cdots \geq y_n$

使得　$x_1+x_2+\cdots+x_n = y_1+y_2+\cdots+y_n$

且　$x_1+x_2+\cdots+x_k \geq y_1+y_2+\cdots+y_k$ for $k=1,2,\ldots,n-1$)

Example: $(3,0,0) \succ (2,1,0)$ (for a, b, c)

我们只能得 $2(a^3+b^3+c^3) \geq a^2b+a^2c+b^2c+b^2a+c^2a+c^2b$

并不能得　$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$

上述情况要用 AM-GM 才成。

所以我们暂时拥有了两个分别应付循环式和对称式的工具。

The quantity $\mathcal{P}(r)$ is called the rth power mean. Note that if we set all the weights equal, that is $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$, then

To save the space, we will introduce three tricks from the pre-pages.

1. product 1

$$\mathcal{P}(r) = \begin{cases} \left(\dfrac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{1/r} & r \neq 0 \\ \sqrt[n]{a_1 a_2 \dots a_n} & r = 0. \end{cases}$$

If it says $abc=1$, we can let $a=\frac{x}{y}$, $b=\frac{y}{z}$, $c=\frac{z}{x}$, w.l.o.g.

Corollary 2.7 (QM-AM-GM-HM theorem)**.** *Let $a_1, \ldots, a_n$ be positive real numbers. Then*

2. Ravi substitution

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}.$$

If a.b.c are three sides of a triangle, we can replace (a,b,c) with $(x+y, y+z, z+x)$

Proof. Set $r \in \{2, 1, 0, -1\}$ we obtain the inequality. □

Here "QM" and "HM" stand for "quadratic mean" and "harmonic mean"

Here is an application of a $\frac{1}{3}$-power mean.

3. Schur inequality ⟶ $\sum_{cyc} a^r(a^2+bc) \geq \sum_{cyc} a^{r+1}(b+c)$

Example 14 (Taiwan TST 2014)**.** Let $a, b, c > 0$. Prove that

when $r=1$, that's the common form we use:

$$3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}.$$

$a^3+b^3+c^3+3abc \geq \sum_{cyc} a^2 b$

Solution 14. By Power Mean with $r = 1$, $s = \frac{1}{3}$, and weights $\frac{1}{9} + \frac{8}{9} = 1$ we have the inequality

$$\left(\frac{1}{9}\sqrt[3]{\frac{a^3+b^3+c^3}{3}} + \frac{8}{9}\sqrt[3]{abc}\right)^3 \leq \frac{1}{9}\left(\frac{a^3+b^3+c^3}{3}\right) + \frac{8}{9}(abc).$$

Thus it is enough to prove $a^3+b^3+c^3+24abc \leq (a+b+c)^3$, which is clear. ■

Example: $(a^2+b^2)(c^2+d^2) \geq (ac+bd)^2$

§2.4.2 Cauchy and Hölder

vector form: $|a| \cdot |b| \geq |a \cdot b|$

We now present Hölder's inequality; we state the two-variable form for concreteness but the obvious generalization to any number of sequences is valid.

Theorem 2.8 (Hölder's inequality)**.** *Let p and q be positive real numbers. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^n a_i\right)^p \left(\sum_{i=1}^n b_i\right)^q \geq \left(\sum_{i=1}^n \sqrt[p+q]{a_i^p b_i^q}\right)^{p+q}.$$

when $p=q=1$, that is Cathy's inequality:

$\sum_{i=1}^n a_i \cdot \sum_{i=1}^n b_i \geq \left(\sum_{i=1}^n \sqrt{a_i b_i}\right)^2$

Solution 16. This time, we use Hölder with slightly changed weights in order to remove the square root:

$$\left(\sum_{\text{cyc}} \frac{a}{\sqrt{b+c}}\right)^2 \left(\sum_{\text{cyc}} a(b+c)\right) \geq (a+b+c)^3.$$

Again it is enough to prove $(a+b+c)^2 \geq \frac{3}{2}\sum_{\text{cyc}} a(b+c)$ which is true by expanding. ∎

Exercise. Show that if one sets $b_i = 1$ for each i, then Hölder's inequality reduces to a power mean inequality with all weights equal.

§2.5 Inequalities in arbitrary functions

Let I be an open interval (for example $I = (0,\infty)$ or $I = (0,1)$) and let $f\colon (u,v) \to \mathbb{R}$ be a function and let $a_1, a_2, \dots, a_n \in (u,v)$. Suppose that we fix $\frac{a_1+a_2+\cdots+a_n}{n} = a$ (if the inequality is homogeneous, we will often insert such a condition) and we want to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$

is at least (or at most) $nf(a)$. In this section we will provide two methods for doing so.

Definition 2.10. We say that function f is **convex** if the second derivative f'' is nonnegative over all of (u,v). Similarly we say it is **concave** if $f''(x) \leq 0$ for all x. Note that f is convex if and only if $-f$ is concave.

let $p>0$, $q>0$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)=\ln(x)$, then $\frac{1}{p}\ln x+\frac{1}{q}\ln y \leq \ln\left(\frac{x}{p}+\frac{y}{q}\right)$

§2.5.i Jensen and Karamata

We have the following analog of AM-GM now.

i.e. $x^{\frac{1}{p}} \cdot y^{\frac{1}{q}} \leq \frac{1}{p}x+\frac{1}{q}y$

Theorem 2.11 (Jensen's inequality)**.** *Let $f\colon I \to \mathbb{R}$ be a convex function. Then for any $a_1, \dots, a_n \in I$ we have*

加权：

$$\frac{f(a_1)+\cdots+f(a_n)}{n} \geq f\left(\frac{a_1+\cdots+a_n}{n}\right).$$

$\frac{1}{p_1}f(x_1)+\cdots+\frac{1}{p_n}f(x_n) \leq f\left(\frac{x_1}{p_1}+\cdots+\frac{x_n}{p_n}\right)$

The reverse inequality holds when f is concave.

Exercise. Show that if one takes $I = (0,\infty)$ and f to be the natural logarithm, then Jensen reduces to AM-GM with all weights equal.

Just as Muirhead is repeated AM-GM, there is an analog of repeated Jensen; however its use is somewhat rarer.

Theorem 2.12 (Karamata's inequality)**.** *Let $f\colon I \to \mathbb{R}$ be convex. Suppose the sequence (x_n) majorizes (y_n), with each x_i and y_i in I. Then*

参见前文"≻"

$$f(x_1)+\cdots+f(x_n) \geq f(y_1)+\cdots+f(y_n).$$

The reverse inequality holds when f is concave.