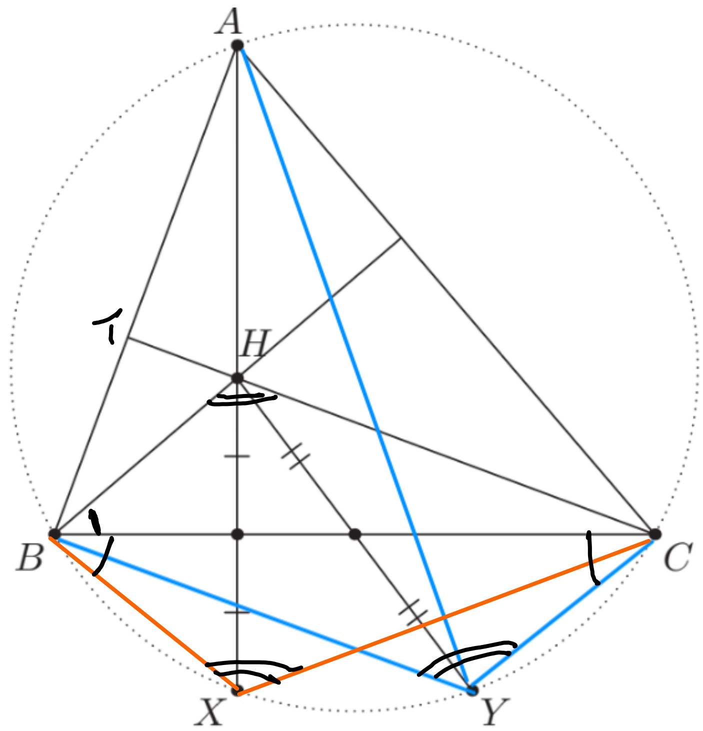


Figure 1.3B. Reflecting the orthocenter. See Lemma 1.17.

Lemma 1.17 (Reflecting the Orthocenter). *Let H be the orthocenter of $\triangle ABC$, as in Figure 1.3B. Let X be the reflection of H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$.*

(a) Show that X lies on (ABC).
(b) Show that $\overline{AY}$ is a diameter of (ABC). **Hint:** 674

首先，易证 $\triangle BHC \cong \triangle BXC$ ∴ $\angle BCH = \angle BCX$
∵ $\angle BCH + \angle XHC = 90^\circ$，$\angle AHT + \angle BAX = 90^\circ$ ∴ $\angle BCH = \angle BAX$
∴ $\angle BCX = \angle BAX$ ∴ X 在 (ABC) 上
其次，易证有▱$BYCH$ ∴ $\angle BCY = \angle HBC = \angle CBX$，$\angle CYB = \angle CHB = \angle CXB$
∴ $\triangle BXY \cong \triangle CYB$ ∴根据圆的对称性，Y 也在 (ABC) 上
∵ $\angle AHT = \angle ABC = \angle XHC$ ∴ $\angle CBY + \angle ABC = 90^\circ$，即 $\angle ABY = 90^\circ$
∴ AY 是 (ABC) 的直径

Lemma 1.18 (The Incenter/Excenter Lemma). *Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then,*

(a) The points I, B, C, and I_A lie on a circle with diameter $\overline{II_A}$ and center L. In particular, $LI = LB = LC = LI_A$.
(b) Rays BI_A and CI_A bisect the exterior angles of $\triangle ABC$.

图见下页。这似乎是很重要的引理（据他自己说）
(a) 显然。根据内心的性质，易得 $LB = LI = LC$，又由对称命题得证。
(b)（证明 BI_A 平分 $\angle MBC$，CI_A 平分 $\angle NCB$）
∵ $\angle IBI_A = 90^\circ$，BI 平分 $\angle ABC$ ∴ $\angle MBI_A = \angle CBI_A$
同理可证 CI_A 平分 $\angle BCN$

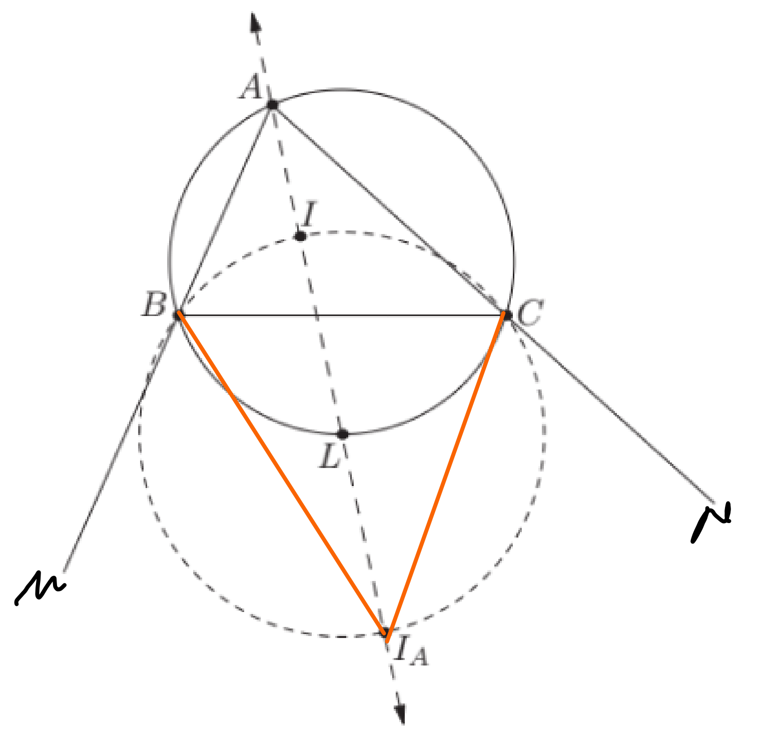


Figure 1.4A. Lemma 1.18, the incenter/excenter lemma.

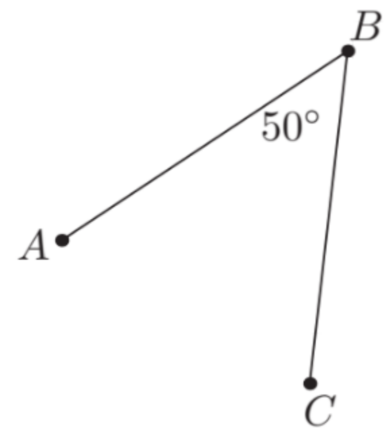
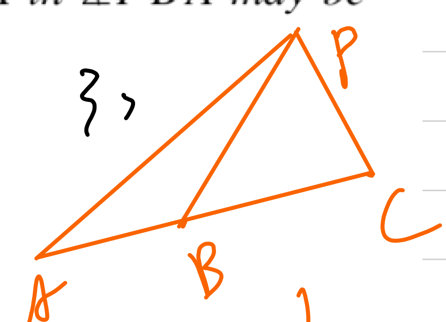
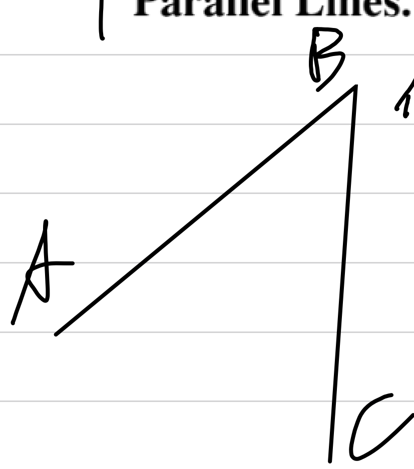


Figure 1.5B. Here, $\measuredangle ABC = 50^\circ$ and $\measuredangle CBA = -50^\circ$.

如上图，出于一些原因，我们开始定义有向角，以下是其性质：

1. **Oblivion.** $\measuredangle APA = 0$.
2. **Anti-Reflexivity.** $\measuredangle ABC = -\measuredangle CBA$.
3. **Replacement.** *$\measuredangle PBA = \measuredangle PBC$ if and only if A, B, C are collinear. (What happens when $P = A$?) Equivalently, if C lies on line BA, then the A in $\measuredangle PBA$ may be replaced by C.*
4. **Right Angles.** *If $\overline{AP} \perp \overline{BP}$, then $\measuredangle APB = \measuredangle BPA = 90^\circ$.*
5. **Directed Angle Addition.** $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.
6. **Triangle Sum.** $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0$.
7. **Isosceles Triangles.** *$AB = AC$ if and only if $\measuredangle ACB = \measuredangle CBA$.*
8. **Inscribed Angle Theorem.** *If (ABC) has center P, then $\measuredangle APB = 2\measuredangle ACB$.*
9. **Parallel Lines.** *If $\overline{AB} \parallel \overline{CD}$, then $\measuredangle ABC + \measuredangle BCD = 0$.*





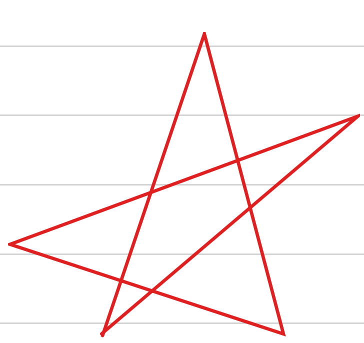
似乎旋转方向很重要

$-30^\circ = 150^\circ = 260^\circ$

$\therefore \measuredangle PBA + \measuredangle CBP = 180^\circ = 0^\circ$

$\measuredangle PBA = -\measuredangle CBP = \measuredangle PBC$

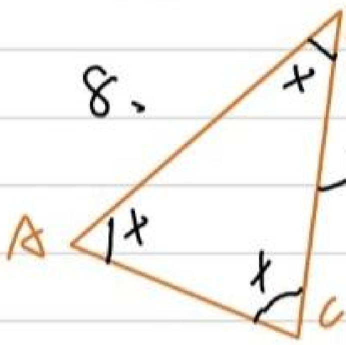
How do we handle this? The solution is to use directed angles mod 180 ∘. Such angles will be denoted with a symbol $\measuredangle$ instead of the standard $\angle$. (This notation is not standard; should you use it on a contest, do not neglect to say so in the opening lines of your solution.)



Here is how it works. First, we consider $\measuredangle$ ABC to be positive if the vertices A, B, C appear in clockwise order, and negative otherwise

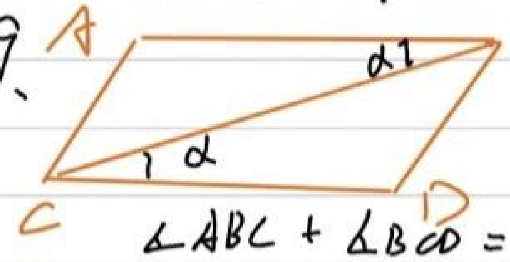
有一说一 这个顺时针方向似乎稍微有点离谱，我猜它的意思差不多等价于这样说：射线AB绕A点顺时针旋转能经过C
不过我还是不能解释大于180°的角怎么来的，几何上似乎很难解释，姑且把它当成运算来看，mod 180° 同余算一个角吧。

8.



→ 三角均为正值，和为180° = 0.

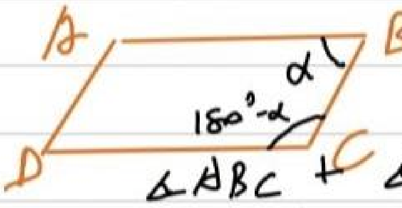
9.



$\measuredangle ABC = \alpha^\circ$

$\measuredangle BCD = -\alpha^\circ$

$\measuredangle ABC + \measuredangle BCD = 0$



$\measuredangle ABC = \alpha^\circ$

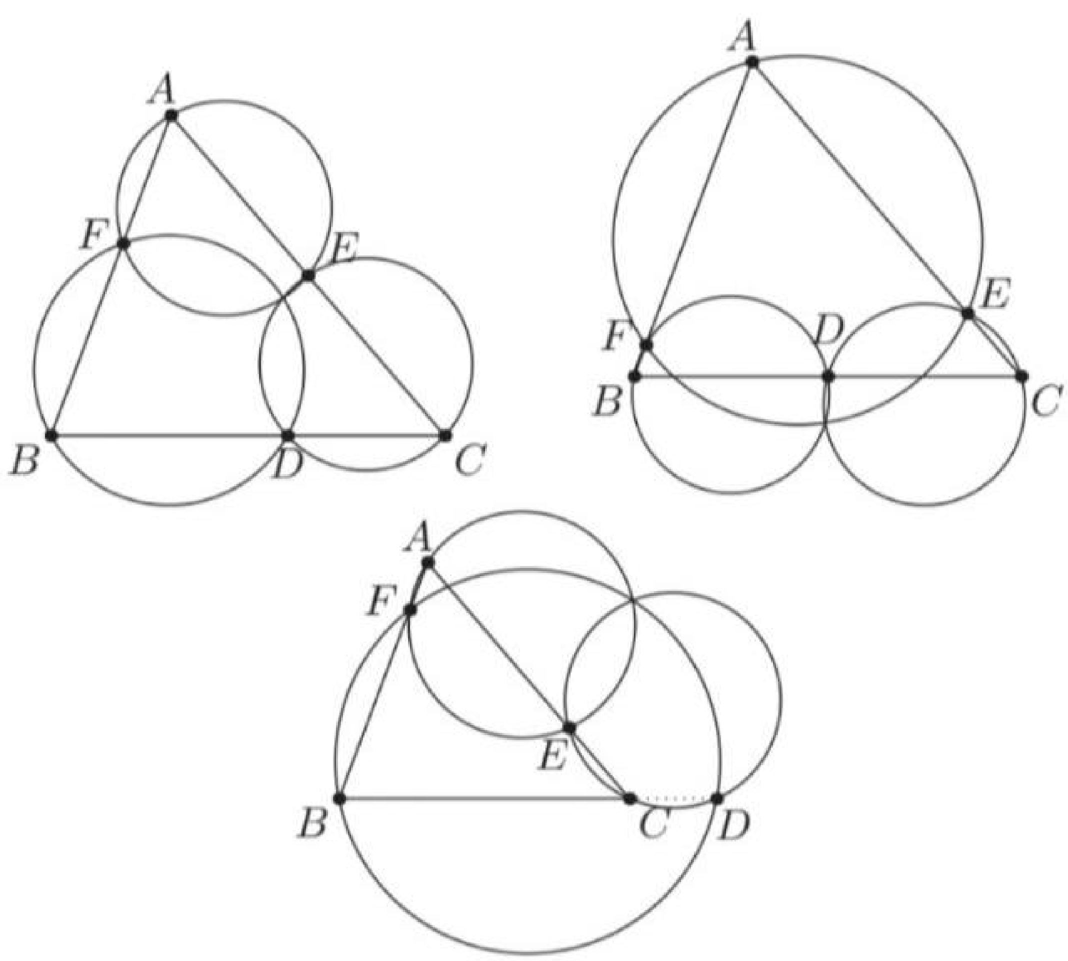
$\measuredangle BCD = 180^\circ - \alpha^\circ$

$\measuredangle ABC + \measuredangle BCD = 180^\circ = 0$

One thing we have to be careful about is that $2\measuredangle ABC = 2\measuredangle XYZ$ does not imply $\measuredangle ABC = \measuredangle XYZ$, because we are taking angles modulo 180°. Hence it does not make sense to take half of a directed angle.

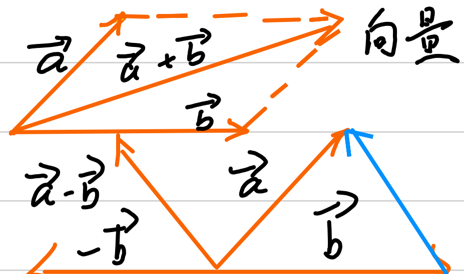
Miquel Point of a Triangle

Lemma 1.27 (Miquel Point of a Triangle). *Points D, E, F lie on lines BC, CA, and AB of $\triangle ABC$, respectively. Then there exists a point lying on all three circles (AEF), (BFD), (CDE).*



It should be clear by looking at the figure on the left that many, many configurations are possible. Trying to handle this with standard angles would be quite messy. Fortunately, we can get them all in one go with directed angles.

好，我们中间插播一下高中课程，今天是向量



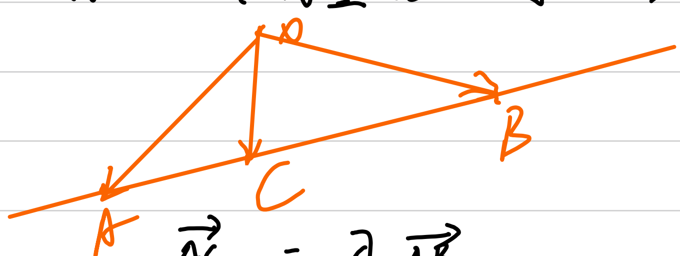
向量相加

向量相减

这老师竟然上课讲群论 23333

三角不等式：$||\vec a|-|\vec b|| \le |\vec a+\vec b| \le |\vec a|+|\vec b|$

1. O、A、B是不同的点，对任意AB(直线)上的点C，向量$\vec{OC}$与$\vec{OA}$和$\vec{OB}$有什么关系



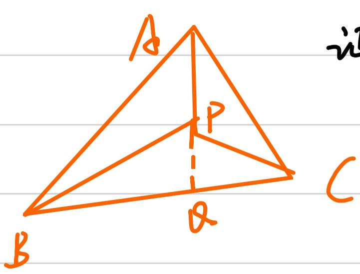
$\vec{AC}=\lambda\vec{AB}$

$\vec{OC}-\vec{OA}=\lambda(\vec{OB}-\vec{OA})$

$\vec{OC}=(1-\lambda)\vec{OA}+\lambda\vec{OB}$

2. 若P为△ABC内一点，则

$S_{\triangle PBC}\cdot\vec{PA}+S_{\triangle PCA}\cdot\vec{PB}+S_{\triangle PAB}\cdot\vec{PC}=0$



证明：$\vec{AP}=\frac{S_{\triangle APB}+S_{\triangle APC}}{S_{\triangle ABC}}\vec{AQ}$

$\vec{AQ}=\frac{S_{\triangle APC}}{S_{\triangle APC}+S_{\triangle APB}}\cdot\vec{AB}+\frac{S_{\triangle APC}}{S_{\triangle APC}+S_{\triangle APB}}\cdot\vec{AC}$

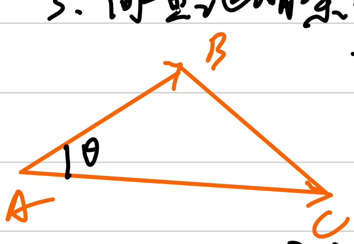
两式代入得 $\vec{AP}=\frac{S_{\triangle APC}}{S_{\triangle ABC}}(\vec{AP}+\vec{PB})+\frac{S_{\triangle APB}}{S_{\triangle ABC}}(\vec{AP}+\vec{PC})$

$\frac{S_{\triangle BPC}}{S_{\triangle ABC}}\cdot\vec{AP}=\frac{S_{\triangle APC}}{S_{\triangle ABC}}\cdot\vec{PB}+\frac{S_{\triangle APB}}{S_{\triangle ABC}}\cdot\vec{PC}$

$\Rightarrow S_{\triangle BPC}\cdot\vec{PA}+S_{\triangle APC}\cdot\vec{PB}+S_{\triangle APB}\cdot\vec{PC}=0$

内积 $\vec a\cdot\vec b=|\vec a||\vec b|\cos\theta$

3. 向量证明余弦定理



$\vec{BC}=\vec{AC}-\vec{AB}$

$|\vec{BC}|^2=(\vec{AC}-\vec{AB})^2$

$=|\vec{AC}|^2-2\vec{AC}\cdot\vec{AB}+|\vec{AB}|^2$

$=|\vec{AC}|^2+|\vec{AB}|^2-2\cos\theta|\vec{AC}||\vec{AB}|$

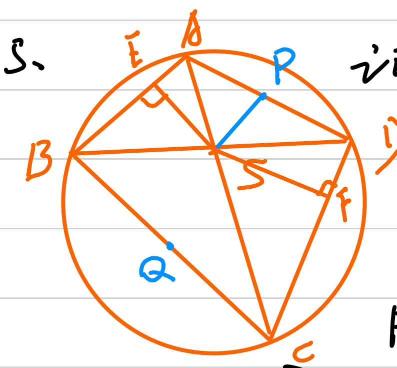
4. 证明：$4\vec a\cdot\vec b=(\vec a+\vec b)^2-(\vec a-\vec b)^2$

证明：$(\vec a+\vec b)^2-(\vec a-\vec b)^2$

$=|\vec a|^2+2\vec a\cdot\vec b+|\vec b|^2-|\vec a|^2+2\vec a\cdot\vec b+|\vec b|^2$

$=4\vec a\cdot\vec b$

5.



证明：$2\vec{SP}=\vec{SA}+\vec{SD}$

∵ SE ⊥ AB

∴ $\vec{SE}\cdot\vec{AE}=0$

∴ $\vec{SE}\cdot\vec{SE}-\vec{SE}\cdot\vec{SA}=0$

同理 $\vec{SF}\cdot\vec{SF}-\vec{SF}\cdot\vec{SD}=0$

∴ $|\vec{PE}|^2-|\vec{PF}|^2=(\vec{SP}-\vec{SE})^2-(\vec{SP}-\vec{SF})^2$

$=|\vec{SE}|^2-2\vec{SP}\cdot\vec{SE}+2\vec{SP}\cdot\vec{SF}-|\vec{SF}|^2$

$=\vec{SE}\cdot\vec{SA}-\vec{SA}\cdot\vec{SE}-\vec{SD}\cdot\vec{SE}+\vec{SA}\cdot\vec{SF}+\vec{SD}\cdot\vec{SF}-\vec{SF}\cdot\vec{SD}$

$=\vec{SA}\cdot\vec{SF}-\vec{SD}\cdot\vec{SE}$

∵ △ASE ∽ △DSF ∴ $\frac{AS}{SE}=\frac{SD}{SF}$，

∠ASE = ∠DSF ∴ ∠ASF = ∠ESD

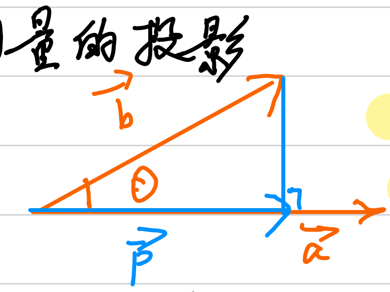
∵ $\vec{SA}\cdot\vec{SF}=|\vec{SA}||\vec{SF}|\cos\angle ASF$

$\vec{SD}\cdot\vec{SE}=|\vec{SD}|\cdot|\vec{SE}|\cos\angle ESD$

∴ $\vec{SA}\cdot\vec{SF}=\vec{SE}\cdot\vec{SD}$ ∴ PE = PF

同理 QE = QF ∴ QP垂直平分EF

向量的投影

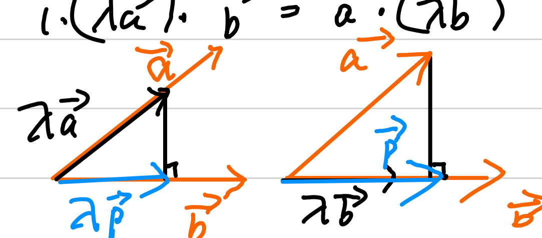


$\vec p=|\vec b|\cos\theta\frac{\vec a}{|\vec a|}$

$\vec a\cdot\vec b=\vec a\cdot\vec p$

内积的性质

1. $(\lambda\vec a)\cdot\vec b=\vec a\cdot(\lambda\vec b)=\lambda(\vec a\cdot\vec b)$

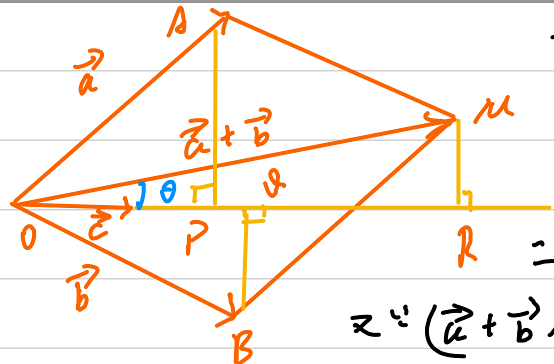


2. 对任意向量$\vec a$，$\vec b$，$\vec c$，有

$(\vec a+\vec b)\cdot\vec c=\vec a\cdot\vec c+\vec b\cdot\vec c$

同样可以用投影来理解

图见下页 →



$\vec{a}\cdot\vec{c} = \vec{c}\cdot\vec{OP}$

$\vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{OQ}$

$\vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

$= |\vec{c}|\cdot(|\vec{OP}| + |\vec{OQ}|)$

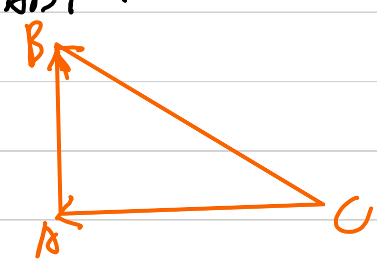
又∵ $(\vec{a}+\vec{b})\cdot\vec{c} = |\vec{OM}||\vec{c}|\cos\theta$

在Rt△MOR中, $OM\cdot\cos\theta = OR$

原命题即转化为 证明 $OP+OQ=OR$

构造全等易证 $RQ=OP$ ∴原命题得证

6. 在△ABC中, $\vec{AB}\cdot\vec{AC} = |\vec{AB}|^2$, 则△ABC的形状为 直角三角形



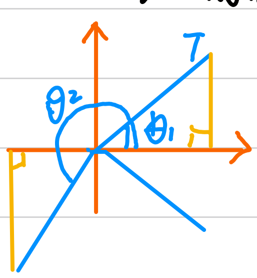
解: $\vec{AB}\cdot\vec{AC} = \vec{AB}\cdot\vec{AB}$

∴ $\vec{AB}\cdot(\vec{AC}-\vec{AB}) = 0$

∴ $\vec{AB}\cdot\vec{BC} = 0$

∴ $AB \perp BC$ ∴ 为Rt△

现在再来补之前漏听的课



$\sin\theta = \frac{y}{OT}$

$\cos\theta = \frac{x}{OT}$

$\tan\theta = \frac{y}{x}$

$\cot\theta = \frac{x}{y}$

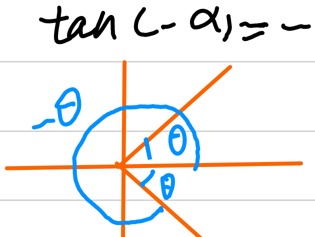
因而我们有以下公式

1. $\sin^2\alpha + \cos^2\alpha = 1$ | 另:
2. $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ | $\sec\alpha = \frac{1}{\cos\alpha}$
3. $\cot\alpha = \frac{1}{\tan\alpha}$ | $\csc\alpha = \frac{1}{\sin\alpha}$

$(\sec^2\alpha = 1+\tan^2\alpha,\ \csc^2\alpha = 1+\cot^2\alpha)$

三角函数诱导公式

1. $\sin(\alpha+2\pi) = \sin\alpha$, $\cos(\alpha+2\pi) = \cos\alpha$
   $\tan(\alpha+2\pi) = \tan\alpha$ (+π的偶数倍不变号)
2. $\sin(\alpha+\pi) = -\sin\alpha$, $\cos(\alpha+\pi) = -\cos\alpha$
   $\tan(\alpha+\pi) = \tan\alpha$ (+π的奇数倍变号(除tan))
3. $\sin(-\alpha) = -\sin\alpha$, $\cos(-\alpha) = \cos\alpha$
   $\tan(-\alpha) = -\tan\alpha$ (角度变号, sin、tan变号)



可画图考虑 $2\pi-\alpha$

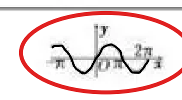
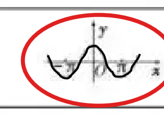
4. $\sin(\pi-\alpha) = \sin\alpha$, $\cos(\pi-\alpha) = -\cos\alpha$
   $\tan(\pi-\alpha) = -\tan\alpha$ (补角[奇数倍π−α]cos变号)
5. $\sin(\frac{\pi}{2}-\alpha) = \cos\alpha$, $\cos(\frac{\pi}{2}-\alpha) = \sin\alpha$
   $\tan(\frac{\pi}{2}-\alpha) = \cot\alpha$ (奇变偶不变)
6. $\sin(\frac{\pi}{2}+\alpha) = \cos\alpha$, $\cos(\frac{\pi}{2}+\alpha) = -\sin\alpha$
   $\tan(\frac{\pi}{2}-\alpha) = -\cot\alpha$ (除tan外和微分一样)

一些弧度与角度的转化

$\pi = 180°$, $360° = 2\pi$, $\frac{\pi}{2} = 90°$

$30° = \frac{\pi}{6}$, $60° = \frac{\pi}{3}$, $45° = \frac{\pi}{4}$

$120° = \frac{2}{3}\pi$, $150° = \frac{5}{6}\pi$, ……

函数	$y=\sin x$	$y=\cos x$	$y=\tan x$
图像			
定义域	R	R	$\{x \mid x\in R$, 且 $x\neq k\pi+\frac{\pi}{2}, k\in Z\}$
值域	$[-1,1]$	$[-1,1]$	R
周期性	2π	2π	π
奇偶性	奇函数	偶函数	奇函数
单调性	$\left[2k\pi-\frac{\pi}{2}, 2k\pi+\frac{\pi}{2}\right]$为增; $\left[2k\pi+\frac{\pi}{2}, 2k\pi+\frac{3\pi}{2}\right]$为减	$[2k\pi, 2k\pi+\pi]$为减; $[2k\pi-\pi, 2k\pi]$为增	$\left(k\pi-\frac{\pi}{2}, k\pi+\frac{\pi}{2}\right)$为增
对称中心	$(k\pi, 0)$	$\left(k\pi+\frac{\pi}{2}, 0\right)$	$\left(\frac{k\pi}{2}, 0\right)$
对称轴	$x=k\pi+\frac{\pi}{2}$	$x=k\pi$	无

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

万能公式: $\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$, $\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

积化和差 & 和差化积

$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$

$\sin A\cos B = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$

$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$