定义352.研究某种随机现象的规律，首先要观察它所有可能的基本结果.例如，将一枚硬币抛掷2次，观察正反面情况我们把对随机现象的实现和对它的观察成为随机试验，简称试验，常用字母E表示，我们感兴趣的是具有以下特点的随机试验:

.试验可以在相同条件下重复进行;

.试验的所有可能结果是明确可知的，并且不止一个;

.每次试验总是恰好出现这些可能结果中的一个，但实现不能确定出现哪-个结果.

我们将随机试验E的每个可能的基本结果称为样本点，用ω表示，称样本点ω构成的集合为样本空间，用Ω表示.在高中阶段，我们一般只讨论Ω为有限集的情况.

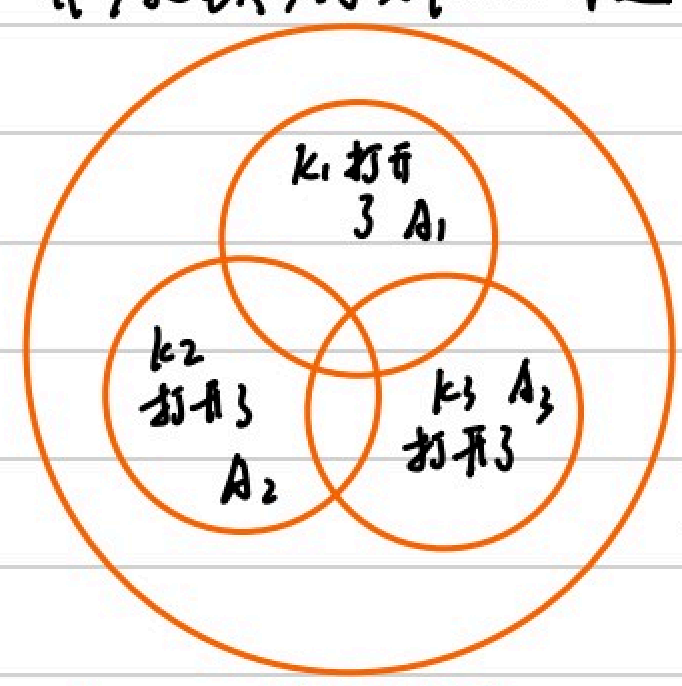
我们将样本空间Ω的子集称为随机事件，简称事件，并把只包含一个样本点的事件称为基本事件.随机事件一般用大写字母A, B,C...表示，在每次试验中，当且仅当A中某个样本点出现时，称为事件A发生

事实上，利用样本空间的子集表示事件，使我们可以利用集合的知识研究随机事件，从而为研究概率的性质和计算等提供有效而简便的方法.我们可以很容易地定义事件的包含，相等，并，交，互斥和互补等关系.

定义353 古典概型 若试验S的样本空间Ω是有限集合，并且Ω中每个样本点发生的可能性相同，我们称S为古典概型

对于古典概型，如果样本点总数为n，事件A包含的样本点个数为m，其中$m、n\in\mathbb{N}$，则$P(A)=\frac{m}{n}$

例(21. PKU) 7把钥匙 7把锁，用7把钥匙分别去开7把锁，问其中3把钥匙k_1, k_2, k_3打不开对应锁的概率?



$|A_1|=|A_2|=|A_3|=6!$

$|A_1\cap A_2|=|A_2\cap A_3|=|A_1\cap A_3|=5!$

$|A_1\cap A_2\cap A_3|=4!$

$\therefore P=\frac{7!-3\times6!+3\times5!-4!}{7!}=\frac{67}{105}$

定义(几何概型) 设样本空间Ω的面积$\mu(\Omega)$是正实数，样本点等可能地落在Ω中(面积相同的区域对应的事件发生概率相同) 对于事件A，$P(A)=\frac{\mu(A)}{\mu(B)}$

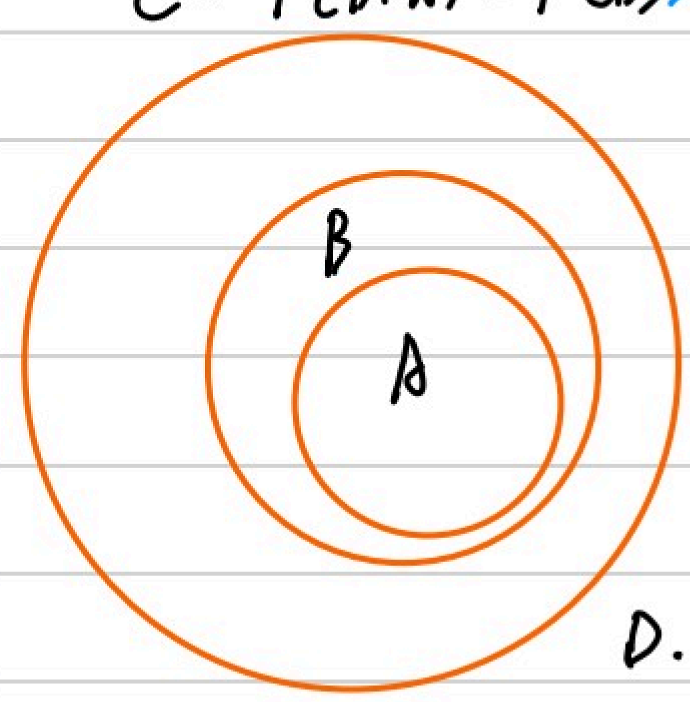
定义(条件概率) 对于任两个事件A、B，已知A发生的情况下B发生的概率叫条件概率，用$P(B|A)$表示。当$P(A)>0$时，定义 $P(B|A)=\frac{P(AB)}{P(A)}$ ($P(AB)$指A.B都发生)

若$P(A)\cdot P(B)=P(AB)$，则称A、B相互独立，有$P(B|A)=P(B)$

例360. 设A.B为随机事件，且$A\subset B$，$0<P(A)<1$，则

A. $P(\overline{A\cdot B})=1-P(B)$ ✗ B. $P(\overline{A}\cdot\overline{B})=1-P(B)$ ✓

C. $P(B|A)=P(B)$ ✗ D. $P(B|\overline{A})=P(B)$ ✗



A. $P(\overline{AB})=P(\overline{A})=1-P(A)$

B. $P(\overline{A}\cdot\overline{B})=P(\overline{B})=1-P(B)$

C. $P(B|A)=P(A)$

D. $P(B|\overline{A})=P(B)-P(A)$

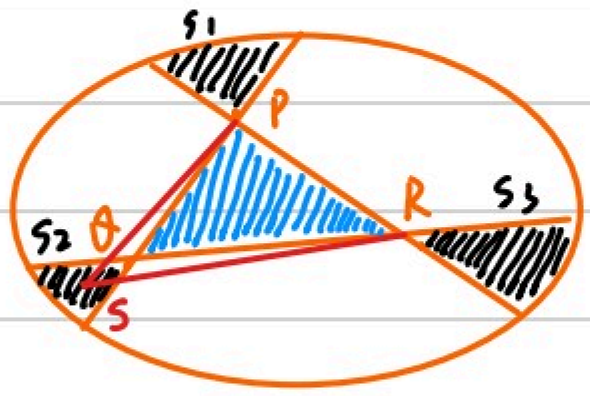
下面是统计，定义同样很tokamology，不打算搬上来了。

例361. 在一组样本数据中，1.2.3.4出现的频率为$p_1、p_2、p_3$、p_4，且$\sum_{i=1}^{4}p_i=1$，下四种情形中，标准差最大的一组为：B

A. $p_1=p_2=0.1$，$p_2=p_3=0.4$ B. $p_1=p_4=0.4$，$p_2=p_3=0.1$

C. $p_1=p_4=0.2$，$p_2=p_3=0.3$ D. $p_1=p_4=0.3$，$p_2=p_3=0.2$

例：☆ 椭圆所围区域中随机取三个点（均匀分布，彼此独立），问 $S_{黑}$ 的期望与 $S_{蓝}$ 的期望的比值是多少？



再随机取一点 S ∴ 四点地位相同

当 S 落入 S_1 时（事件A），P在 $\triangle SQR$ 内，

当 S 落入 S_2 时（B），Q在 $\triangle SPR$ 内，

当 S 落入 S_3 时（C），R在 $\triangle SPQ$ 内

当 S 落入 $\triangle PQR$ 时（D），S 在 $\triangle PQR$ 内

∵ 四点地位相同 ∴ $P(A) = P(B) = P(C) = P(D)$

∴ $S_1 = S_2 = S_3 = S_{\triangle PQR}$

∴ $S_{黑} : S_{蓝} = 3 : 1$

We know that in most inequalities with a constraint such as $abc = 1$ the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ simplifies the solution (don't kid yourself, not all problems of this type become easier!). The use of substitutions is far from being specific to inequalities; there are many other similar substitutions that usually make life easier. For instance, have you ever thought of other conditions such as

$$xyz = x + y + z + 2; \quad xy + yz + zx + 2xyz = 1; \quad x^2 + y^2 + z^2 + 2xyz = 1$$

or $x^2 + y^2 + z^2 = xyz + 4$? The purpose of this chapter is to present some of the most classical substitutions of this kind and their applications.

开始看 Problems from THE BOOK，不知道到底难不难呢。

You will be probably surprised (unless you already know it...) when finding out that the condition $xyz = x + y + z + 2$ together with $x, y, z > 0$ implies the existence of positive real numbers a, b, c such that

$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c}.$$

$x+y+z+2 = xyz \Rightarrow \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$ （从左到右是怎么想到的） 令 $a = \frac{1}{1+x}$，$b = \frac{1}{1+y}$，$c = \frac{1}{1+z}$

$a+b+c=1$ ∴ $x = \frac{1-a}{a} = \frac{b+c}{a}$ 同理可得 y、z

Then $xy + yz + zx + 2xyz = 1$ $(x, y, z > 0)$？

Note that $xy + yz + zx + 2xyz = 1 \Leftrightarrow \frac{1}{z} + \frac{1}{x} + \frac{1}{y} + 2 = \frac{1}{xyz}$, so we can easily get the result:

$$x = \frac{a}{b+c}, \quad y = \frac{b}{a+c}, \quad z = \frac{c}{a+b}$$

Now, let us take a closer look at the other substitutions mentioned at the beginning of the chapter, namely $x^2 + y^2 + z^2 + 2xyz = 1$ and $x^2 + y^2 + z^2 = xyz + 4$. Let us begin with the following question, which can be considered an exercise, too: consider three real numbers a, b, c such that $abc = 1$ and let

$$\boxed{x = a + \frac{1}{a}, \quad y = b + \frac{1}{b}, \quad z = c + \frac{1}{c} \qquad (1.1)}$$

The question is to find an algebraic relation between x, y, z, independent of a, b, c.

Note that

$$
\begin{aligned}
xyz &= \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \\
&= \left(a^2+\frac{1}{a^2}\right)+\left(b^2+\frac{1}{b^2}\right)+\left(c^2+\frac{1}{c^2}\right)+2 \\
&= (x^2-2)+(y^2-2)+(z^2-2)+2.
\end{aligned}
$$

Thus,

$$x^2+y^2+z^2-xyz=4 \qquad (1.2)$$

Because $\left|a+\frac{1}{a}\right| \geq 2$ for all real numbers a, it is clear that not every triple (x, y, z) satisfying (1.2) is of the form (1.1).

So when $x=y=z=1$ the left side doesn't equal to the right side. What we need is $\min\{|x|, |y|, |z|\} \geq 2$

Also, it's suffices to assume only that $\max(|x|, |y|, |z|) > 2$.. Indeed, we may assume that $|x| > 2$.

Thus we suppose that $x = u + \frac{1}{u}$ while $u \in \mathbb{R}$.

Now, let us regard (1.2) as a quadratic equation with respect to z. Because the discriminant is nonnegative, it follows that $(x^2-4)(y^2-4) \geq 0$. But since $|x| > 2$, we find that $y^2 \geq 4$ and so there exist a non-zero real number v for which $y = v + \dfrac{1}{v}$. How do we find the corresponding z? Simply by solving the second degree equation. We find two solutions:

$(u+\frac{1}{u})(v+\frac{1}{v})z$

$= (u+\frac{1}{u})^2 + (v+\frac{1}{v})^2 + z^2 - 4$

$\therefore z_1 + z_2 = (u+\frac{1}{u})(v+\frac{1}{v})$

$z_1 z_2 = (u+\frac{1}{u})^2 + (v+\frac{1}{v})^2 - 4$

$$z_1 = uv + \frac{1}{uv}, \quad z_2 = \frac{u}{v} + \frac{v}{u}$$

and now we are almost done. If $z = uv + \dfrac{1}{uv}$ we take $(a, b, c) = \left(u, v, \dfrac{1}{uv}\right)$ and if $z = \dfrac{u}{v} + \dfrac{v}{u}$, then we take $(a, b, c) = \left(\dfrac{1}{u}, v, \dfrac{u}{v}\right)$.

Inspired by the previous equation, let us consider another one,

$$\boxed{x^2+y^2+z^2+xyz=4 \qquad (1.3)}$$

where $x, y, z > 0$. We will prove that the set of solutions of this equation is the set of triples $(2\cos A, 2\cos B, 2\cos C)$, where A, B, C are the angles of an acute triangle. First, let us prove that all these triples are solutions.

Proof: We have $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$, then we'll have the result.

C

$\cos^2 A + \cos^2 B + \cos^2(\pi - A - B) + 2\cos A \cos B \cos(\pi - A - B)$

$= \cos^2 A + \cos^2 B + \cos^2(A+B) - (\cos(A-B) + \cos(A+B))\cos(A+B)$

$= \cos^2 A + \cos^2 B - \cos(A-B)\cos(A+B)$

$= \cos^2 A + \cos^2 B - \frac{1}{2}\cos 2B - \frac{1}{2}\cos 2A$

$= \cos^2 B - \frac{1}{2}\cos^2 B + \frac{1}{2}\sin^2 B - \frac{1}{2}\cos^2 A + \frac{1}{2}\sin^2 A + \cos^2 A$

$= \frac{1}{2}(\cos^2 B + \sin^2 B) + \frac{1}{2}(\cos^2 A + \sin^2 A) = \frac{1}{2} + \frac{1}{2} = 1$)

Let us summarize: we have seen some nice substitutions, with even nicer proofs, but we still have not seen any applications. We will see them in a moment... and there are quite a few problems that can be solved by using these "tricks". First, an easy and classical problem, due to Nesbitt . It has so many extensions and generalizations that we must discuss it first.

Example 1. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

for all $a, b, c > 0$.

证明：令 $x = \frac{a}{b+c}$，$y = \frac{b}{a+c}$，$z = \frac{c}{a+b}$ ∴ $xy + yz + zx + 2xyz = 1$ 若 $x+y+z < \frac{3}{2}$，∵ $xy + zy + xz \le \frac{(x+y+z)^2}{3}$ ($\Leftarrow xy + yz + zx \le x^2+y^2+z^2 \Leftarrow 2xy + 2yz + 2zx \le (x^2+y^2) + (y^2+z^2) + (z^2+x^2)$) ∴ $xy + zy + xz < (\frac{3}{2})^2/3 = \frac{3}{4}$ 又∵ $xyz \le (\frac{x+y+z}{3})^3$ ($\sqrt[n]{\prod_{i=1}^{n} a_i} \le \frac{\sum_{i=1}^{n} a_i}{n}$，均值不等式) ∴ $2xyz < \frac{1}{4}$

∴ $2xyz + xy + yz + zx < 1$，矛盾 ∴ 原命题成立

Example 2. Let $x, y, z > 0$ be such that $xy + yz + zx + 2xyz = 1$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 4(x + y + z).$$

[Mircea Lascu]

证明：∵ $xy + yz + zx + 2xyz = 1$ ∴ 使用我们的代换，化为 $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 4(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b})$，又∵ 我们有 $\frac{4a}{b+c} \leq \frac{a}{b} + \frac{a}{c}$, $\frac{4b}{c+a} \leq \frac{b}{c} + \frac{b}{a}$, $\frac{4c}{a+b} \leq \frac{c}{a} + \frac{c}{b}$. ∴ 可证明。

$\frac{a}{b} + \frac{a}{c} - \frac{4a}{b+c} = \frac{a(b+c)}{bc} - \frac{4a}{b+c} = \frac{a(b+c)^2 - 4a}{bc(b+c)} = \frac{a(b+c+2)(b+c-2)}{bc(b+c)}$ 下面分类讨论可证明 / 突然发现就是均值不等式

$\frac{b+c}{2} \geq \frac{2}{\frac{1}{b}+\frac{1}{c}} \Rightarrow \frac{2}{b+c} \leq \frac{1}{2b} + \frac{1}{2c} \Rightarrow \frac{4a}{b+c} \leq \frac{a}{b} + \frac{a}{c}$

Example 3. Prove that in any acute-angled triangle ABC the following inequality holds

$$\cos^2 A \cos^2 B + \cos^2 B \cos^2 C + \cos^2 C \cos^2 A$$
$$\leq \frac{1}{4}(\cos^2 A + \cos^2 B + \cos^2 C).$$

[Titu Andreescu]

证明：观察得 $\frac{\cos A \cos B}{\cos C} + \frac{\cos B \cos C}{\cos A} + \frac{\cos C \cos A}{\cos B} \leq \frac{1}{4}(\frac{\cos A}{\cos B \cos C} + \frac{\cos B}{\cos A \cos C} + \frac{\cos C}{\cos A \cos B})$

令 $x = \frac{\cos A \cos B}{\cos C}$，$y = \frac{\cos B \cos C}{\cos A}$，$z = \frac{\cos C \cos A}{\cos B}$

∴ 原式化为 $4(x+y+z) \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$，这恰恰是上一题的不等式，所以我们仅需证明

$xy + yz + xz + 2xyz = 1$，这意味着 $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ (已证明)

Example 4. Find all polynomials $f(x, y, z)$ with real coefficients such that

$$f\left(a+\frac{1}{a}, b+\frac{1}{b}, c+\frac{1}{c}\right)=0$$

whenever $abc = 1$.

[Gabriel Dospinescu]

解：很显然所有可被 $x^2+y^2+z^2-xyz-4$ 整除的式子都满足条件，但是我们很难说明所有 f 都是这种形式。因此我们使用多项式长除法。设有 $g(x,y,z)$，$h(y,z)$，$k(y,z)$ 使得：

$$f(x, y, z)=(x^2+y^2+z^2-xyz-4) g(x, y, z)+x h(y, z)+k(y, z)$$

由命题要求，$(a+\frac{1}{a})h(b+\frac{1}{b}, c+\frac{1}{c})+k(b+\frac{1}{b}, c+\frac{1}{c})=0$，when $abc=1$

we take two numbers x, y such that $\min\{|x|,|y|\}>2$ and we write $x=b+\dfrac{1}{b}$, $y=c+\dfrac{1}{c}$ with $b=\dfrac{x+\sqrt{x^2-4}}{2}$, $c=\dfrac{y+\sqrt{y^2-4}}{2}$.
Then it is easy to compute $a+\dfrac{1}{a}$. It is exactly

$$x y+\sqrt{(x^2-4)(y^2-4)}.$$

So, we have found that

$$\left(x y+\sqrt{(x^2-4)(y^2-4)}\right) h(x, y)+k(x, y)=0$$

whenever $\min\{|x|,|y|\}>2$. And now? The last relation suggests that we should prove that for each y with $|y|>2$, the function $x \rightarrow \sqrt{x^2-4}$ is not rational, that is, there are not polynomials p, q such that $\sqrt{x^2-4}=\dfrac{p(x)}{q(x)}$.

问题保留

But this is easy 有重根 because if such polynomials existed, than each zero of x^2-4 should have even multiplicity, which is not the case. Consequently, for each y with $|y|>2$ we have $h(x, y)=k(x, y)=0$ for all x. But this means that $h(x, y)=k(x, y)=0$ for all x, y, that is our polynomial is divisible by $x^2+y^2+z^2-xyz-4$.