Informal Notes on Mathematics

2022.07.28

定义 208 (空间向量). 1. 在空间中, 我们把具有大小和方向的量叫做空间向量, 空间向量的大小叫做空间向量的长度或模.

2. 特别地, 我们将长度为 0 的向量叫做零向量, 记为 $\overrightarrow{0}$, 将长度为 1 的向量叫做单位向量. 与向量 $\overrightarrow{a}$ 长度相等而方向相反的向量叫做 $\overrightarrow{a}$ 的反向量, 记为 $-\overrightarrow{a}$.

3. 如果表示若干空间向量的有向线段所在的直线互相平行或重合, 那么这些向量叫做共线向量或平行向量. 特别地, 我们规定: 零向量与任意向量平行.

4. 空间相量的加法, 数乘和点乘同平面向量.

定理 209 (共线向量). 两个非零的空间向量 $\overrightarrow{a}, \overrightarrow{b}$ 共线的充要条件是存在一个非零常数 λ 满足 $\overrightarrow{a} = \lambda\overrightarrow{b}$.

定理 210 (共面向量). 若 $\overrightarrow{a}, \overrightarrow{b}$ 是两个不共线的向量, 那么向量 $\overrightarrow{p}$ 与 $\overrightarrow{a}, \overrightarrow{b}$ 共面的充要条件为存在两个常数 λ, μ 满足 $\overrightarrow{p} = \lambda\overrightarrow{a} + \mu\overrightarrow{b}$.

线性无关.

定理 211 (空间向量分解). 取定三个不共面的向量 $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$, 那么对任意空间向量 $\overrightarrow{p}$ 都存在唯一的三个实数 x, y, z 满足 $\overrightarrow{p} = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$.

①线性无关.

生成空间

spanning

span

如果相关 ⇒ 共面

② 向量个数与维数相同

$\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ 是一组基

单位

定义 212 (空间的标准正交基). 在空间中的任意三个互相垂直向量 $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ 都称作一组标准正交基, 由上一个定理, 我们可以通过这个正交基写出向量的坐标.

性质 213 (向量加法, 数乘, 点乘的坐标公式). 固定一个标准正交基 $\{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$, 对空间中任意两个向量 $\overrightarrow{a} = (x_a, y_a, z_a)$, $\overrightarrow{b} = (x_b, y_b, z_b)$ 和任意实数 λ 都有

1. $\overrightarrow{a} + \overrightarrow{b} = (x_a + x_b, y_a + y_b, z_a + z_b)$.
2. $\lambda\overrightarrow{a} = (\lambda x_a, \lambda y_a, \lambda z_a)$.
3. $\overrightarrow{a} \cdot \overrightarrow{b} = x_a x_b + y_a y_b + z_a z_b$. (标准正交基)
4. $\cos\left(\overrightarrow{a}, \overrightarrow{b}\right) = \frac{x_a x_b + y_a y_b + z_a z_b}{\sqrt{x_a^2+y_a^2+z_a^2} \cdot \sqrt{x_b^2+y_b^2+z_b^2}}$.

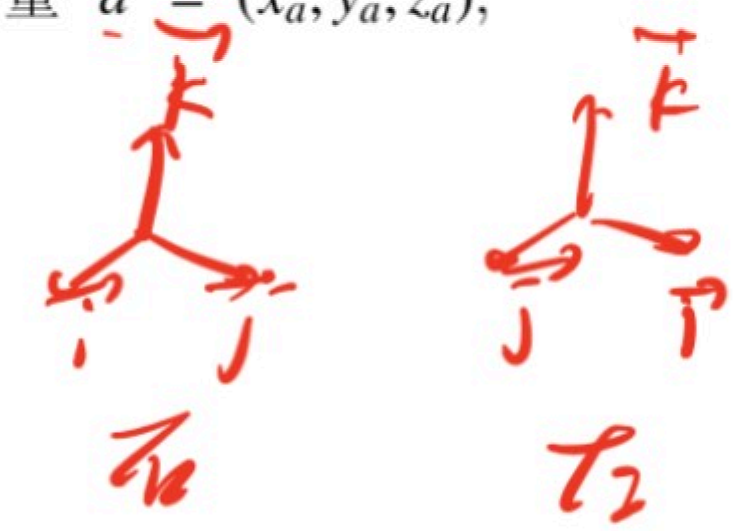
右手系.

定义 214 (叉乘). 固定一个标准正交基 $\{\vec{i},\vec{j},\vec{k}\}$, 对空间中任意两个向量 $\vec{a}=(x_a,y_a,z_a)$, $\vec{b}=(x_b,y_b,z_b)$, 我们定义一个向量

$$\vec{a}\times\vec{b}=(y_az_b-y_bz_a,\ z_ax_b-z_bx_a,\ x_ay_b-x_by_a)$$

称为 $\vec{a},\vec{b}$ 的叉乘.

$\vec{a}\times\vec{b}$ 与 $\vec{a},\vec{b}$ 垂直.



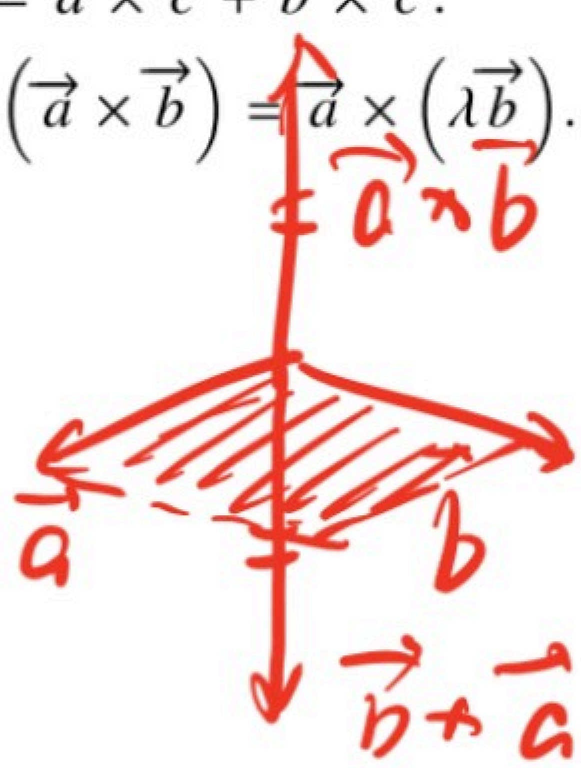
性质 215 (叉乘的基本性质). 对任意空间向量 $\vec{a},\vec{b},\vec{c}$ 和实数 λ 都有:

1. $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$.
2. $\vec{a}\times\vec{a}=\vec{a}\times\vec{0}=\vec{0}$.
3. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$.
4. $\vec{a}\times\vec{b}$ 与 $\vec{a}$ 和 $\vec{b}$ 垂直.
5. $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.
6. $\left(\vec{a}+\vec{b}\right)\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.
7. $\left(\lambda\vec{a}\right)\times\vec{b}=\lambda\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(\lambda\vec{b}\right)$.

$(\vec{a},\vec{b},\vec{a}\times\vec{b})$ 右手系.

$\vec{i}\times\vec{j}=\vec{k}$ 右.

$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\langle\vec{a},\vec{b}\rangle$



今天来看 Vakil 的 The Rising Sea.

Our general approach will be as follows. I will try to tell you what you need to know, and no more. (This I promise: if I use the word "topoi", you can shoot me.) I will begin by telling you things you already know, and describing what is essential about the examples, in a way that we can abstract a more general definition. We will then see this definition in less familiar settings, and get comfortable with using it to solve problems and prove theorems.

For example, we will define the notion of *product* of schemes. We could just give a definition of product, but then you should want to know why this precise definition deserves the name of "product". As a motivation, we revisit the notion of product in a situation we know well: (the category of) sets. One way to define the product of sets U and V is as the set of ordered pairs $\{(u,v)\ :\ u\in U, v\in V\}$. But someone from a different mathematical culture might reasonably define it as the set of symbols $\{{}^{u}_{v}:\ u\in U, v\in V\}$. These notions are "obviously the same". Better: there is "an obvious bijection between the two".

This can be made precise by giving a better definition of product, in terms of a *universal property*. Given two sets M and N, a product is a set P, along with maps $\mu\colon P \to M$ and $\nu\colon P \to N$, such that for *any set* P' *with maps* $\mu'\colon P' \to M$ *and* $\nu'\colon P' \to N$, these maps must factor *uniquely* through P:

(1.1.0.1)
$$\begin{tikzcd} P' \arrow[dr, dotted, "\exists!"] \arrow[drr, "\nu'"] \arrow[ddr, "\mu'"'] & & \\ & P \arrow[r, "\nu"] \arrow[d, "\mu"] & N \\ & M & \end{tikzcd}$$

(The symbol $\exists$ means "there exists", and the symbol ! here means "unique".) Thus a **product** is a *diagram*

$$\begin{tikzcd} P \arrow[r, "\nu"] \arrow[d, "\mu"'] & N \\ M & \end{tikzcd}$$

and not just a set P, although the maps μ and ν are often left 隐式的 implicit.

某种意义上这种定义挺符合直觉的。我感觉独立性其实也就暗示了有序性，不过这样定义确实回避了 product 只是某个特定集合的情况。现在还想不到什么很脱离直觉的例子。

This definition agrees with the traditional definition, with one twist: there isn't just a single product; but any two products come with a *unique* isomorphism between them. In other words, the product is unique up to unique isomorphism. Here is why: if you have a product

$$\begin{tikzcd} P_1 \arrow[r, "\nu_1"] \arrow[d, "\mu_1"'] & N \\ M & \end{tikzcd}$$

and I have a product

$$\begin{tikzcd} P_2 \arrow[r, "\nu_2"] \arrow[d, "\mu_2"'] & N \\ M & \end{tikzcd}$$

因此由 universal property 可知：存在唯一的 f, g，使得以下图交换

$$\begin{tikzcd}
P_2 \arrow[dr, "f"] \arrow[drr, "\nu_2"] \arrow[ddr, "\mu_2"'] & & \\
& P_1 \arrow[r, "\nu_1"] \arrow[d, "\mu_1"] & M \\
& N &
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1 \arrow[dr, "g"] \arrow[drr, "\nu_1"] \arrow[ddr, "\mu_1"'] & & \\
& P_2 \arrow[r, "\nu_2"] \arrow[d, "\mu_2"] & M \\
& N &
\end{tikzcd}$$

同样，让 P_2 同时充当 P, P' 的角色，则存在唯一的映射，我们称之为 id_{P_2}，使得以下图交换：

就是 identity map

$$\begin{tikzcd}
P_2 \arrow[dr, "id_{P_2}"] \arrow[drr, "\nu_2"] \arrow[ddr, "\mu_2"'] & & \\
& P_2 \arrow[r, "\nu_2"] \arrow[d, "\mu_2"] & M \\
& N &
\end{tikzcd}$$

很容易看出：
$$\begin{tikzcd}
P_2 \arrow[r, "f"] & P_1 \arrow[r, "g"] & P_2 \arrow[ll, bend right, "id_{P_2}"']
\end{tikzcd}$$
，因此有 $f \circ g = id_{P_2}$ 同理 $g \circ f = id_{P_1}$

∴ 由 universal property 得出的 f, g 为双射，∴ 我们可以唯一地命名 $M \times N$

This definition has the advantage that it works in many circumstances, and once we define categories, we will soon see that the above argument applies verbatim in any category to show that products, if they exist, are unique up to unique isomorphism. Even if you haven't seen the definition of category before, you can verify that this agrees with your notion of product in some category that you have seen before (such as the category of vector spaces, where the maps are taken to be linear maps; or the category of differentiable manifolds, where the maps are taken to be *submersions*, i.e., differentiable maps whose differential is everywhere surjective). Я изучила инженсон я нет щуовруод лищзирэ !

A **category** consists of a collection of **objects**, and for each pair of objects, a set of **morphisms** (or **arrows**) between them. (For experts: technically, this is the definition of a *locally small category*. In the correct definition, the morphisms need only form a class, not necessarily a set, but see Caution 0.3.1.) Morphisms are often informally called **maps**. The collection of objects of a category $\mathscr{C}$ is often denoted $\mathrm{obj}(\mathscr{C})$, but we will usually denote the collection also by $\mathscr{C}$. If $A, B \in \mathscr{C}$, then the set of morphisms from A to B is denoted $\mathrm{Mor}(A, B)$. A morphism is often written $f\colon A \to B$, and A is said to be the **source** of f, and B the **target** of f. (Of course, $\mathrm{Mor}(A, B)$ is taken to be disjoint from $\mathrm{Mor}(A', B')$ unless $A = A'$ and $B = B'$.)

Morphisms compose as expected: there is a composition $\mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)$, and if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then their composition is denoted $g \circ f$. Composition is associative: if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$. For each object $A \in \mathscr{C}$, there is always an **identity morphism** $\mathrm{id}_A : A \to A$, such that when you (left- or right-)compose a morphism with the identity, you get the same morphism. More precisely, for any morphisms $f\colon A \to B$ and $g\colon B \to C$, $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$. (If you wish, you may check that "identity morphisms are unique": there is only one morphism deserving the name id_A.) This ends the definition of a category.

We have a notion of **isomorphism** between two objects of a category (a morphism $f: A \to B$ such that there exists some — necessarily unique — morphism $g: B \to A$, where $f \circ g$ and $g \circ f$ are the identity on B and A respectively), and a notion of **automorphism** of an object (an isomorphism of the object with itself).

1.2.2. *Example.* The prototypical example to keep in mind is the category of sets, denoted *Sets*. The objects are sets, and the morphisms are maps of sets. (Because Russell's paradox shows that there is no set of all sets, we did not say earlier that there is a set of all objects. But as stated in §0.3, we are deliberately omitting all set-theoretic issues.)

1.2.3. *Example.* Another good example is the category Vec_k of vector spaces over a given field k. The objects are k-vector spaces, and the morphisms are linear transformations. (What are the isomorphisms?)

1.2.A. UNIMPORTANT EXERCISE. A category in which each morphism is an isomorphism is called a **groupoid**. (This notion is not important in what we will discuss. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)
(a) A perverse definition of a *group* is: a groupoid with one object. Make sense of this.
(b) Describe a groupoid that is not a group.

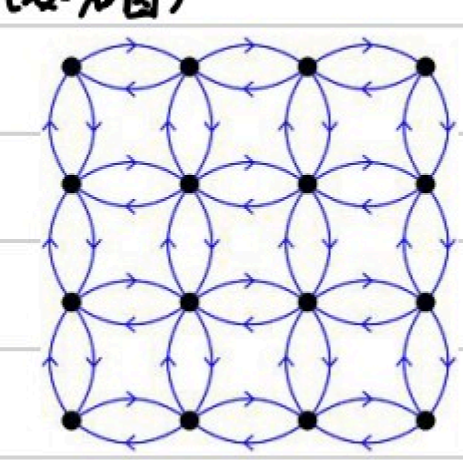
Хаоузяхуо，妙中典了。Я каньдао йле юнуюйg exaмпле в MathOverflow：

3阶魔方的操作构成一个群，15块拼图的操作构成一个groupoid.（如右图）

如右图，展示了空格块可能在的16个位置及其移动方法

两个操作被视为相同当且仅当它们的起点和终点相同，

且对数字的影响相同。

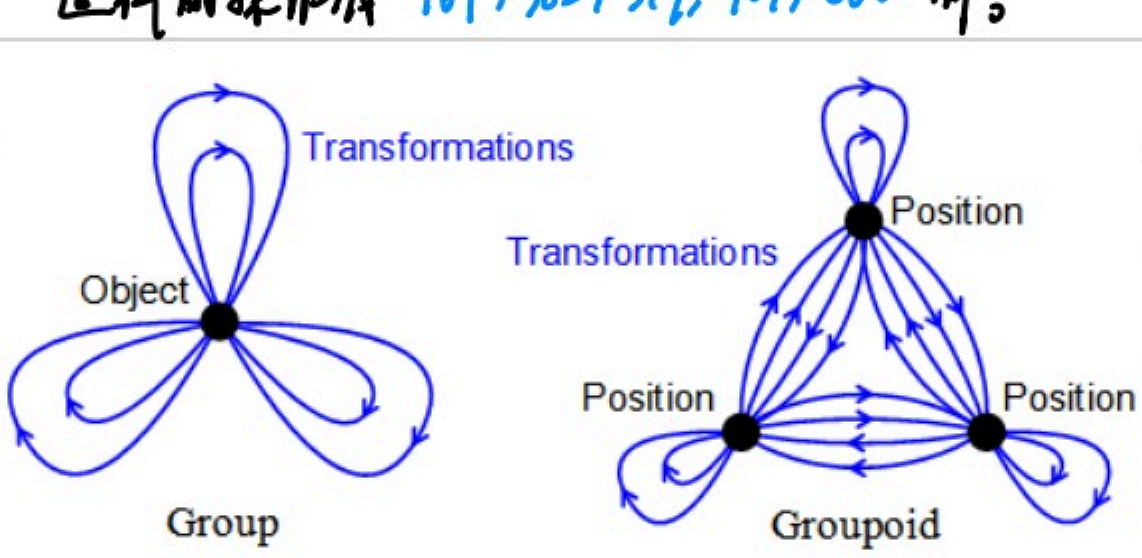


1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

和魔方不同的是，15拼图的操作并不总是能复合的。

操作需要满足：(1) 其复合仍是一个操作（封闭性） (2) 操作是可逆的（可逆性）

这样的操作有 167,382,319,104,000 种。



一类情况是只有一个obj的groupoid是group，并非每两个obj之间都存在映射的groupoid是一族群。

另一类情况，所有等价关系都是一个groupoid. 对于 $a, b \in obj(R)$ 最多有一个映射 $a \to b$。

这个回答的作者是 Tom Leinster 诶。真是 carpaucun！