Informal Notes on Mathematics

2022.08.13

看一点复分析。好久没做数学笔记了。

So what about complex functions? If you consider them as functions $\mathbb{R}^2 \to \mathbb{R}^2$, you now have the interesting property that you can integrate along things that are not line segments: you can write integrals across curves in the plane. But $\mathbb{C}$ has something more: it is a *field*, so you can *multiply* and *divide* two complex numbers.

So we restrict our attention to differentiable functions called *holomorphic functions*. It turns out that the multiplication on $\mathbb{C}$ makes all the difference. The primary theme in what follows is that holomorphic functions are *really, really nice*, and that knowing tiny amounts of data about the function can determine all its values.

The two main highlights of this chapter, from which all other results are more or less corollaries:

- Contour integrals of loops are always zero.
- A holomorphic function is essentially given by its Taylor series; in particular, single-differentiable implies infinitely differentiable. Thus, holomorphic functions behave quite like polynomials.

Some of the resulting corollaries:

- It'll turn out that knowing the values of a holomorphic function on the boundary of the unit circle will tell you the values in its interior.
- Knowing the values of the function at $1, \frac{1}{2}, \frac{1}{3}, \dots$ are enough to determine the whole function!
- Bounded holomorphic functions $\mathbb{C} \to \mathbb{C}$ must be constant
- And more. . .

全纯函数的确有"窥一斑以知全貌"的特性。复函数的导数我们之前已经有所了解了。(柯西-黎曼方程稍微放后)

下面我们来看Contour积分。

§31.3 Contour integrals

Prototypical example for this section: $\oint_\gamma z^m \, dz$ *around the unit circle.*

In the real line we knew how to integrate a function across a line segment $[a, b]$: essentially, we'd "follow along" the line segment adding up the values of f we see to get some area. Unlike in the real line, in the complex plane we have the power to integrate over arbitrary paths: for example, we might compute an integral around a unit circle. A contour integral lets us formalize this.

First of all, if $f : \mathbb{R} \to \mathbb{C}$ and $f(t) = u(t) + iv(t)$ for $u, v \in \mathbb{R}$, we can define an integral $\int_a^b$ by just adding the real and imaginary parts:

$$\int_a^b f(t)\, dt = \left(\int_a^b u(t)\, dt\right) + i \left(\int_a^b v(t)\, dt\right).$$

Now let $\alpha : [a, b] \to \mathbb{C}$ be a path, thought of as a complex differentiable[2] function. Such a path is called a **contour**, and we define its **contour integral** by

$$\oint_\alpha f(z)\, dz = \int_a^b f(\alpha(t)) \cdot \alpha'(t)\, dt.$$

You can almost think of this as a u-substitution (which is where the α' comes from). In particular, it turns out this integral does not depend on how α is "parametrized" (参数化的): a circle given by

$$[0, 2\pi] \to \mathbb{C} : t \mapsto e^{it}$$

and another circle given by

$$[0, 1] \to \mathbb{C} : t \mapsto e^{2\pi i t}$$

[2] This isn't entirely correct here: you want the path α to be continuous and mostly differentiable, but you allow a finite number of points to have "sharp bends"; in other words, you can consider paths which are combinations of n smooth pieces. But for this we also require that α has "bounded length".

and yet another circle given by

$$[0, 1] \to \mathbb{C} : t \mapsto e^{2\pi i t^5}$$

will all give the same contour integral, because the paths they represent have the same geometric description: "run around the unit circle once".

In what follows I try to use α for general contours and γ in the special case of loops.

Let's see an example of a contour integral.

Wait, wait, wait. 让我们先回顾一下实积分，然后再考虑怎么直观认识复积分。

和处理微分一样, 我们现在也从比较熟知的实函数积分的概念开始. 这个过程的历史根源, 以及使之可视化的主要手段, 都是计算函数图像下的面积问题.

我们先用矩形来逼近所求面积. 见图 8-1, 把积分区间分成 n 个线段 Δ_i（用作矩形之底, 其长度也用 Δ_i 来表示）, 从每个线段中随机地取一点 x_i, 并以函数在此点上方的高度 $f(x_i)$ 作为此矩形的高. 于是每个矩形的面积就是 $f(x_i)\Delta_i$, 而 f 下方总的矩形面积逼近值就是

$$R \equiv \sum_{i=1}^{n} f(x_i)\Delta_i. \tag{8.1}$$

R 就称为**黎曼和**. 最后, 要求 n 趋于无穷且每一个 Δ_i 都趋于零, 就得到所求的面积.

你可能还模模糊糊地记得哪位教授以前对你讲过 (8.1), 甚至说不定你还自己用这个方法做过计算. 但是, 一旦注意上了微积分的基本定理, 肯定就会很快忘记

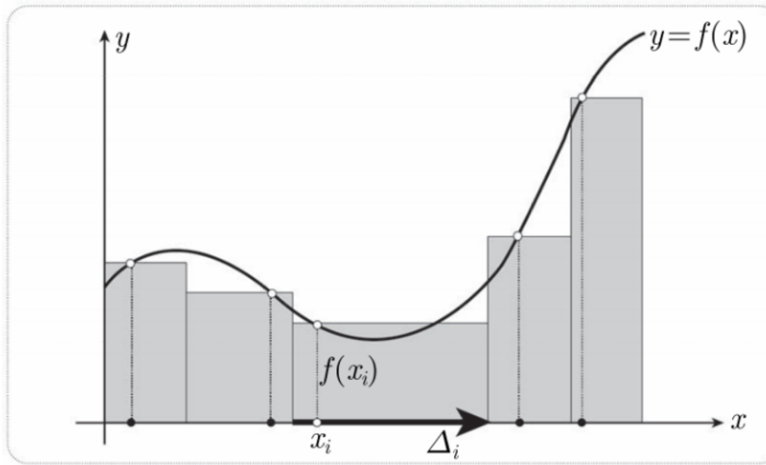


图 8-1

8.2.2 梯形法则

望文生义, 我们现在用梯形而不是矩形来逼近面积. 为方便起见, 取所有的 Δ_i 均有相同长度, 虽然这并非完全必要. 从图 8-2 上就可以清楚地看到, 不太大的 n 就可以给出相当精确的估计. 因为图 8-2 的形状和图 8-1 不同, 相应的梯形公式 (我们不打算把它写出来) 与 (8.1) 也大不相同. 然而, 如果我们打算继续使用 (8.1), 也不难找到一个极其模仿梯形和的黎曼和, 从而保持梯形和的精度.

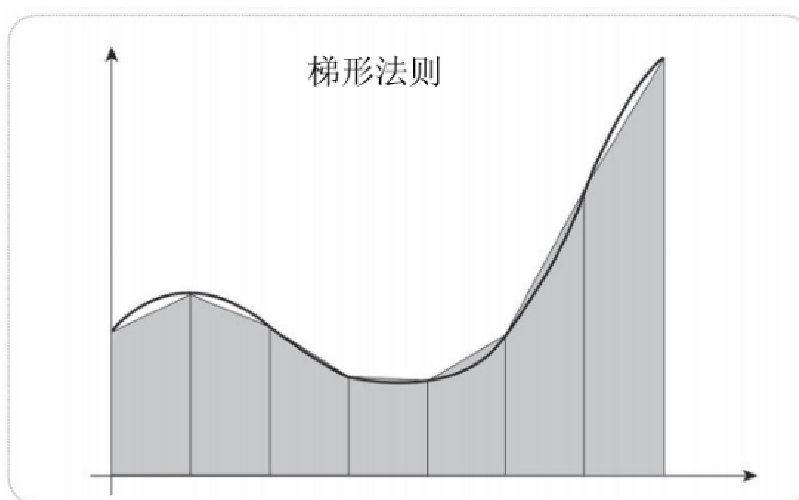
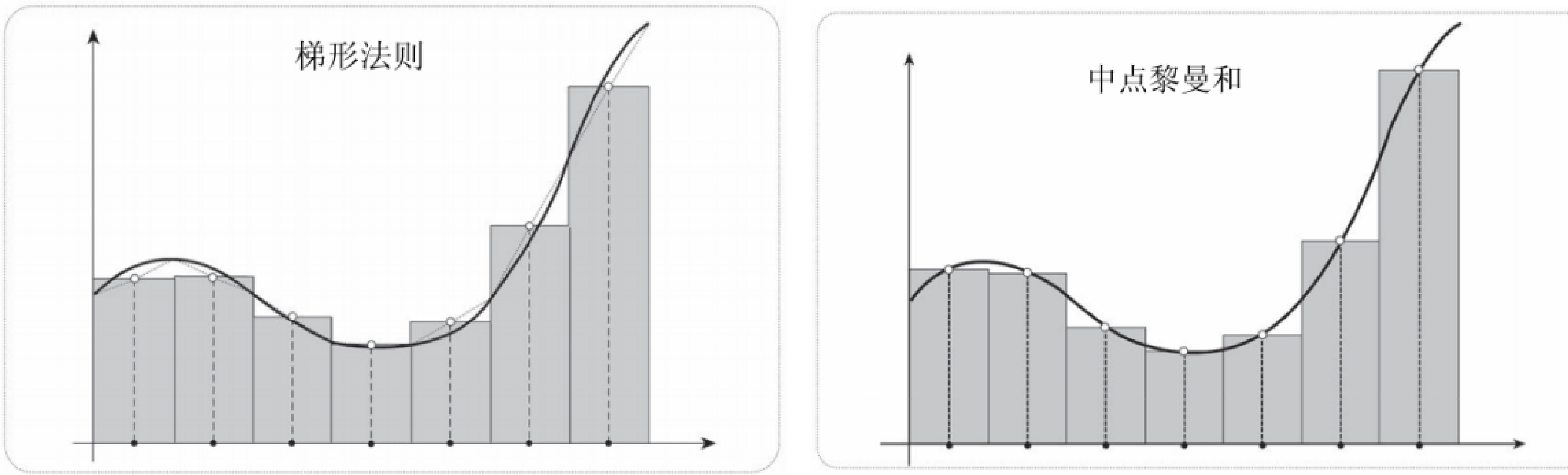


图 8-2

首先注意图 8-2 上画有阴影的梯形估计与图 8-3 的矩形估计是完全一样的, 在图 8-3 中的矩形高取为 Δ_i 的中点处的弦的高度. 然后, 为了恢复到黎曼和, 我们又把弦的高度代以在那个中点处曲线的高度. 见图 8-4. 换言之, 只要取 x_i 为 Δ_i的中点, 就能对不太大的 n 得到一个黎曼和 (8.1) 而给出积分的精确估计, 我们称这个黎曼和为**中点黎曼和**, 记作 R_M.



8.3.1 复黎曼和

在实积分情况下, 我们由一个很明确的几何目标（求面积）开始, 然后发明了积分作为达到目的的手段. 在复情况下, 我们则把这个过程颠倒过来, 也就是说, 首先 "盲目地" 推广实积分（通过黎曼和）, 只是到后来才会问问自己究竟创造出什么了. 首先, 在本章, 我们将找出一种方法把积分 "画" 成单个复数. 然后在第 11 章, 我们将从一种完全不同的观点看出, 一个积分的实部和虚部分别有生动的几何（和物理）意义. 如果想要事前就猜想出有关的几何实体, 然后再去发明复积分作为找出这些几何实体的合适的工具, 那就需要在想象力方面有一次惊人的巨大飞跃——历史上没有发生过这样的事. 稍想一下就会发现, 在微分方面情况也是类似的, 那里从斜率概念开始, 然后经过一个开始时的盲目外推过程, 最终得到了一个很不相同的（但是直观性并不稍次的）伸扭的思想.

考虑图 8-7, 为了将复映射 $f(z)$ 从 a 到 b 积分, 我们需指定一条连接两点的曲线并沿着它做积分. 这条曲线（记为 K）现在就起积分区间的作用, 和图 8-1 一样, 我们将它分为小段 Δ_i, 这里为方便起见, 设它们均有相同长度. 这里与图 8-1 的区别在于, 现在这些小段的方向并不一致. 为了构造黎曼和, 我们在 K 的每一小段中取 z_i 点然后做乘积 $f(z_i)\Delta_i$ 的和. 最后, 让小段的数目增加而使 Δ_i 越来越紧贴着 K, 这时黎曼和将趋向一极限值（只要映射是连续的）, 这个极限值就是复积分的定义, 记作

$$\int_K f(z)\mathrm{d}z.$$

和实积分一样, 我们可以不取极限而得积分的精确估计, 只要取 z_i 为 K 的各小段的**中点**, 而不是随机地取 z_i. 事实上, 在图 8-7 中我们就是这样做的. 我们再一次把这个特别准确的黎曼和记作 R_M.

为了进而理解 R_M 的几何意义, 请看图 8-8. 图上画出了 K 在映射 $z \mapsto w = f(z)$ 下的象, 特别是标出了图 8-7 中的各点 z_i 之象 w_i. R_M 中相应的项是 $\widetilde{\Delta}_i \equiv w_i\Delta_i$, 我们把它看作 w_i"作用于"Δ_i 所得的向量, 即将 Δ_i 之长放大 $|w_i|$ 倍并旋转一个角 $\arg(w_i)$.

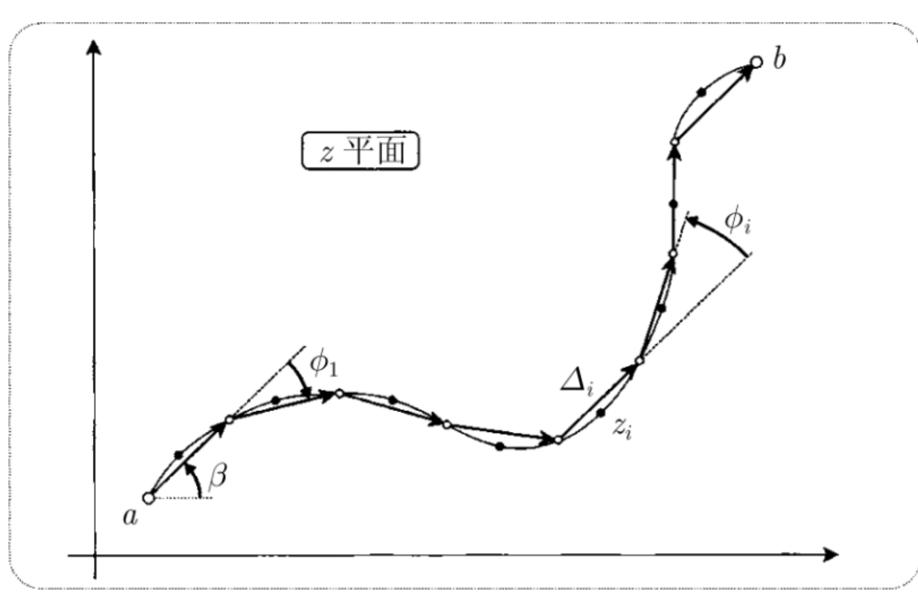
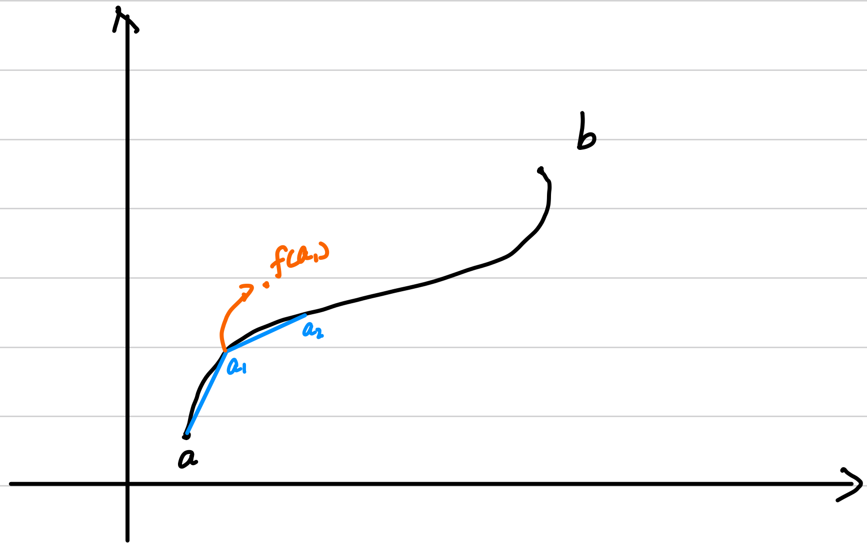


图 8-7

所以说复积分确实和路径相关，但不能完全理解成求路径长。(?)



下面我们看一个例子：

Theorem 31.3.1

Take $\gamma : [0, 2\pi] \to \mathbb{C}$ to be the unit circle specified by

$$t \mapsto e^{it}.$$

Then for any integer m, we have

$$\oint_\gamma z^m \, dz = \begin{cases} 2\pi i & m = -1 \\ 0 & \text{otherwise} \end{cases}$$

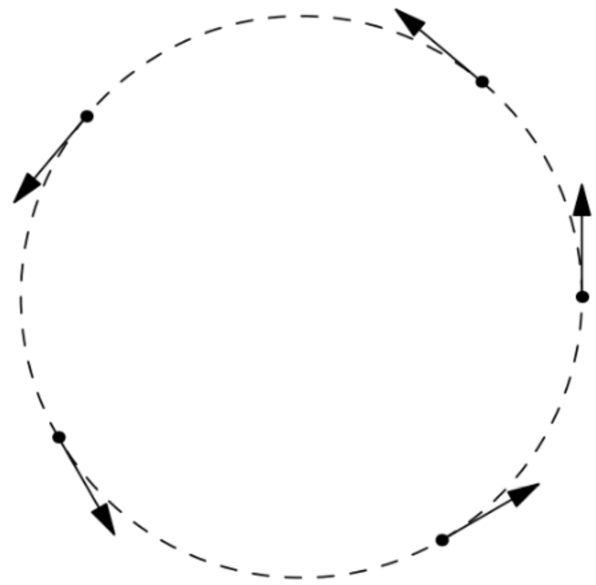
证明：按照公式来即可：$\because (e^{it})' = ie^{it}$

$$\therefore \oint_\gamma z^m \, dz = \int_0^{2\pi} (e^{it})^m \cdot ie^{it} \, dt = \int_0^{2\pi} i (e^{it})^{1+m} \, dt$$

$$= i\int_0^{2\pi} e^{(1+m)it} \, dt = -\int_0^{2\pi} \sin[(1+m)t] \, dt + i\int_0^{2\pi} \cos[(1+m)t] \, dt$$

This is now an elementary calculus question. One can see that this equals $2\pi i$ if $m = -1$ and otherwise the integrals vanish. $\square$

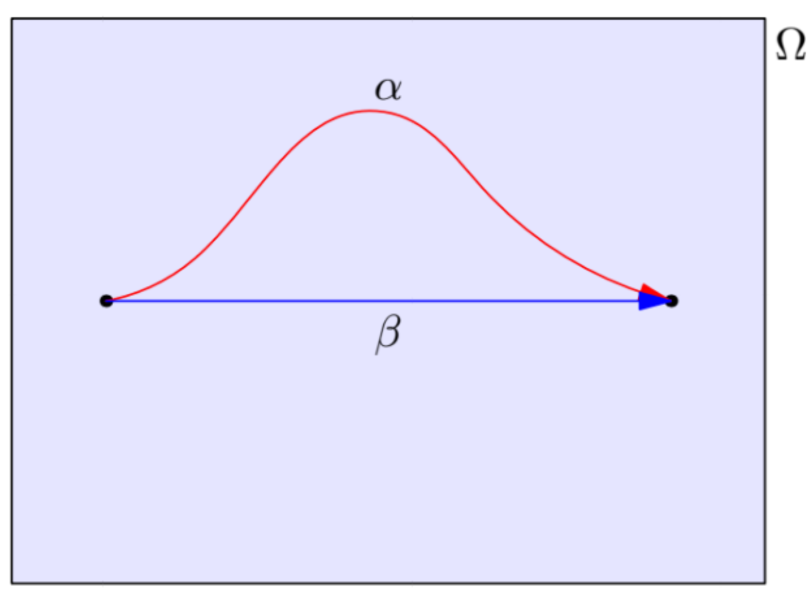
Let me try to explain why this intuitively ought to be true for $m = 0$. In that case we have $\oint_\gamma 1 \, dz$. So as the integral walks around the unit circle, it "sums up" all the tangent vectors at every point (that's the direction it's walking in), multiplied by 1. And given the nice symmetry of the circle, it should come as no surprise that everything cancels out. The theorem says that even if we multiply by z^m for $m \neq -1$, we get the same cancellation.



Definition 31.3.2. Given $\alpha : [0,1] \to \mathbb{C}$, we denote by $\overline{\alpha}$ the "backwards" contour $\overline{\alpha}(t) = \alpha(1-t)$.

> **Question 31.3.3.** What's the relation between $\oint_\alpha f\,dz$ and $\oint_{\overline{\alpha}} f\,dz$? Prove it.

Let $\Omega \subseteq \mathbb{C}$ be simply connected (for example, $\Omega = \mathbb{C}$), and consider two paths α, β with the same start and end points.



What's the relation between $\oint_\alpha f(z)\,dz$ and $\oint_\beta f(z)\,dz$? You might expect there to be some relation between them, considering that the space Ω is simply connected. But you probably wouldn't expect there to be *much* of a relation.

仅对于全纯函数，这两个积分相等。我扔掉了那个非全纯的反例。现在的心态不适合看数学。

Theorem 31.4.2 (Cauchy-Goursat theorem)

Let γ be a loop, and $f : \Omega \to \mathbb{C}$ a holomorphic function where Ω is open in $\mathbb{C}$ and simply connected. Then

$$\oint_\gamma f(z)\,dz = 0.$$