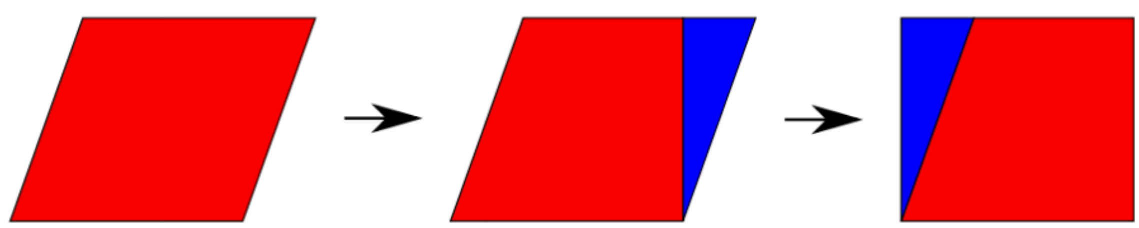
Tarski's circle-squaring problem

主要资料：A. Marks, S. Unger. arXiv: 1612.05833 & the talk

In 1925, Tarski posed the problem of whether a disk and square of the same area in the plane are equidecomposable by isometries [T]. That is, can a disk be partitioned into finitely many pieces which can be rearranged by isometries to partition a square of the same area? This problem became known as Tarski's circle squaring problem. In contrast to the Banach-Tarski paradox in $\mathbb{R}^3$, a theorem of Tarski (see [W]) implies that any two Lebesgue measurable sets in $\mathbb{R}^2$ that are equidecomposable by isometries must have the same Lebesgue measure, even when the pieces used in the equidecomposition are allowed to be nonmeasurable. Thus, the requirement that the circle and the square have the same area is necessary.

Dissection congruence (分割同等) (合同)

Two polygons are **dissection congruent** if we can cut the first into finitely many polygons which we can rearrange to get the second (ignoring boundaries). This idea dates back to Euclid.



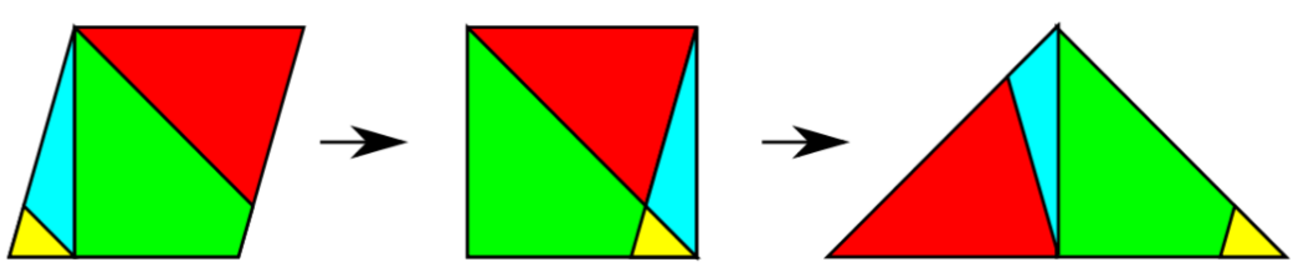
Area$=bh$

If two polygons are dissection congruent, they have the same area.

Theorem (Wallace-Bolyai-Gerwein, 1807, 1833)

Any two polygons of the same area are dissection congruent.

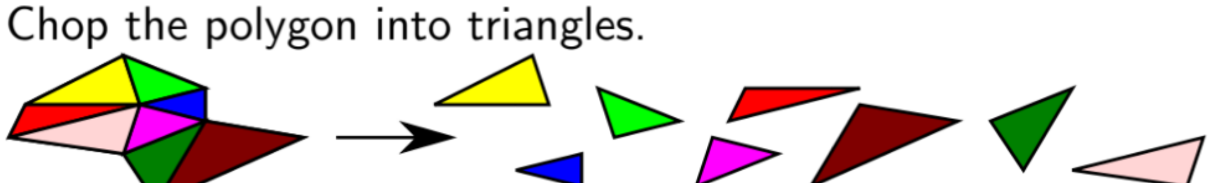
P is dissection congruent to Q and Q is dissection congruent to R implies P is dissection congruent to R.



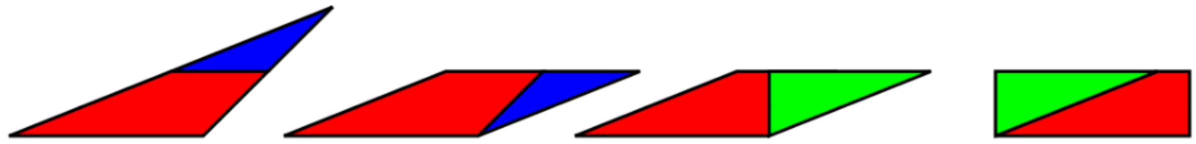
P Q R

Proving the Wallace-Bolyai-Gerwein theorem

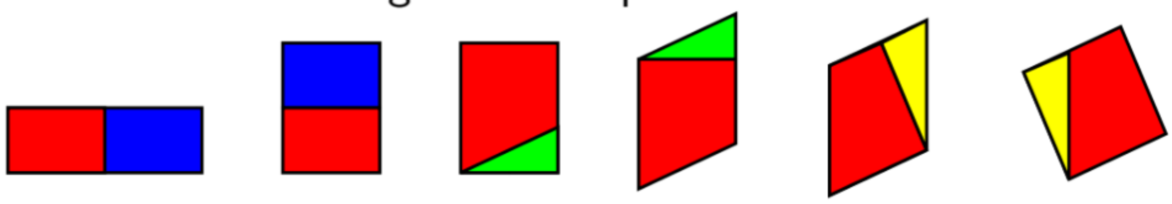
To show a polygon is dissection congruent to a square:
Chop the polygon into triangles.



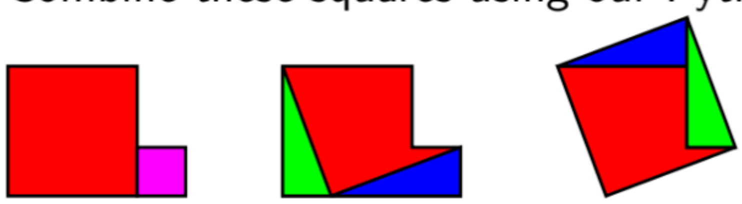
Dissect each triangle into a parallelogram and then a rectangle



Dissect each rectangle into a square



Combine these squares using our Pythagorean proof.



□

Hilbert's third problem

Hilbert's third problem: show any two polytopes of the same volume in three dimensions are dissection congruent.

Theorem (Dehn, 1902)

A cube and a regular tetrahedron are not dissection congruent.

Indeed, if P is a polyhedron with edge lengths ℓ_i and edge dihedral angles θ_i, then the **Dehn invariant**

$$\sum_i \ell_i \otimes \theta_i$$ 张量积

(taking values in the tensor product $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}$) is an invariant of dissection congruence.

Theorem (Sydler, 1965)

Two polyhedra are dissection congruent if and only if they have the same volume and Dehn invariant.

这里突然提到了张量积，确实让人有点为难。有时间试着做一份相关笔记吧。

The existence of Vitali sets implies that for all $n \geq 1$, there is no extension of Lebesgue measure to the full powerset $P(\mathbb{R}^n)$ which is

1. invariant under isometries, and
2. countably additive.

Dropping condition (1) leads to real valued measurable cardinals.

If we weaken condition (2) to finite additivity, there is no such measure for $n \geq 3$ because of the Banach-Tarski paradox (1924). In contrast, for $n \leq 2$ there are finitely additive isometry invariant measures extending Lebesgue measure on $\mathbb{R}^n$. These are called **Banach measures**.

这个悖论主要是为了表明选择公理可推出极诡异的结果，稍后再提。

(The difference hinges on the fact that if $n \geq 3$, the isometry group of $\mathbb{R}^n$ contains a free group on two generators. If $n \leq 2$ it does not.)

Tarski's circle squaring problem

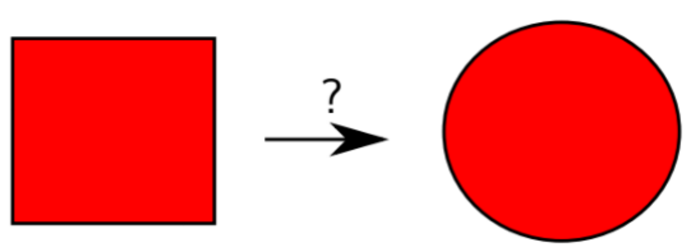
等可分解的

$A, B \subseteq \mathbb{R}^n$ are **equidecomposable** if A can be partitioned into finitely many pieces which can be rearranged by isometries to partition B.

Central question: what is the relationship between equidecomposability and measure?

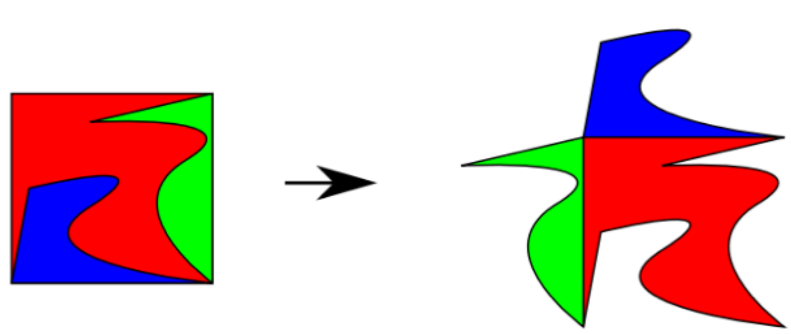
Question (Tarski's circle squaring problem, 1925)

Are a disc and square in $\mathbb{R}^2$ (necessarily of the same area) equidecomposable?



The disc and square must have the same area because of the existence of Banach measures.

A and B are **scissors congruent** if A can be cut into finitely pieces–each of which is homeomorphic to a disc and bounded by a curve of finite length–which can be rearranged to form B (ignoring boundaries).



In scissors congruence, any time a section of convex circular perimeter is created or destroyed it cancels with a corresponding pieces of concave circular perimeter. So

$$\text{convex circular perimeter} - \text{concave circular perimeter}$$

is an invariant of scissors congruence.

Corollary (Dubins-Hirsch-Karush, 1964)

A square and disc are not scissors congruent.

直观地来说，应该是指一段凸曲边一定对应另一块的凹曲边，这对于 square-disc 不可能

下面的部分开始有趣了： Laczkovich's solution

Theorem (Laczkovich, 1990 (AC))

Tarski's circle squaring problem has a positive answer!

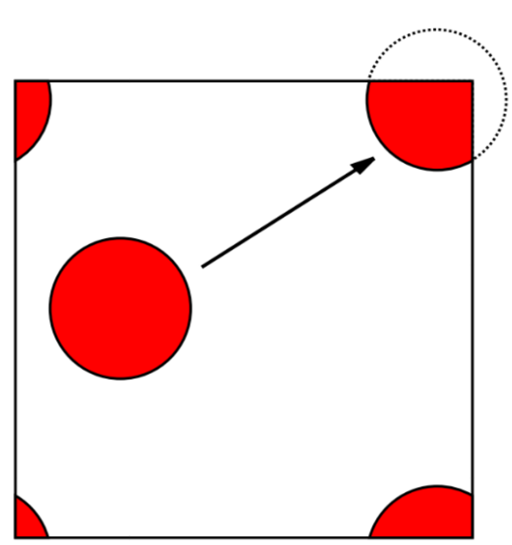
More generally,

Theorem (Laczkovich, 1992 (AC))

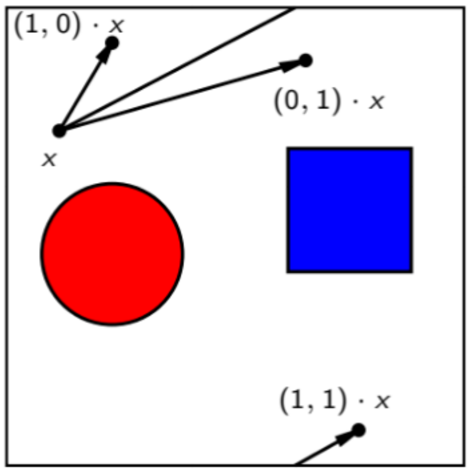
If $A, B \subseteq \mathbb{R}^k$ are bounded sets with the same positive Lebesgue measure whose boundaries have upper Minkowski dimension less than k, then A and B are equidecomposable.

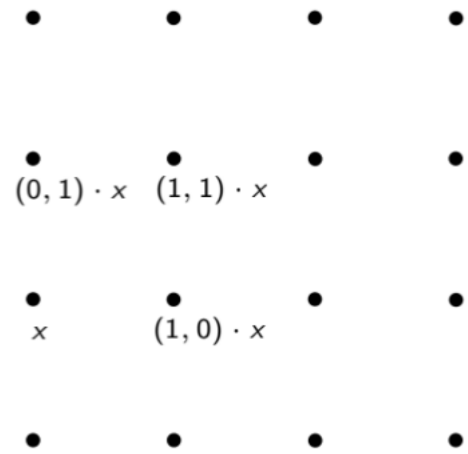
First idea: **Work in the torus**

Fix sets A, B. Scale and translate A and B so that they lie in $[0,1)^k$ which we identify with the k-torus $\mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k$. Then A and B are equidecomposable by translations as subsets of $\mathbb{T}^k$ iff they are equidecomposable by translations in $\mathbb{R}^k$.
(Though perhaps using more pieces).



Use random translations

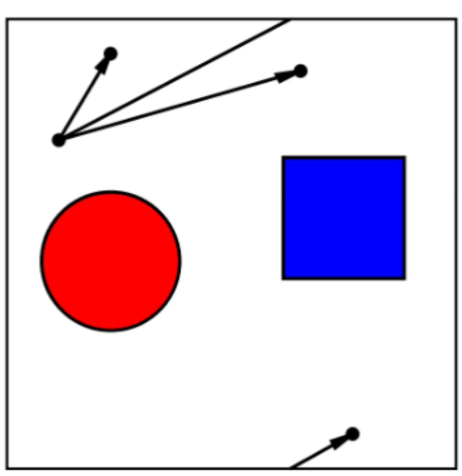


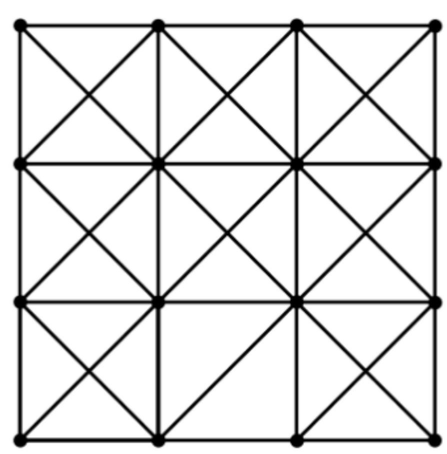


Fix a sufficiently large d, and random $u_1, \ldots, u_d \in \mathbb{T}^k$. Obtain a random action of $\mathbb{Z}^d$ on $\mathbb{T}^k$ by translations:

$$(n_1, \ldots, n_d) \cdot x = n_1 u_1 + \ldots + n_d u_d + x$$

This action is almost surely free. We can visualize each orbit as a copy of $\mathbb{Z}^d$.

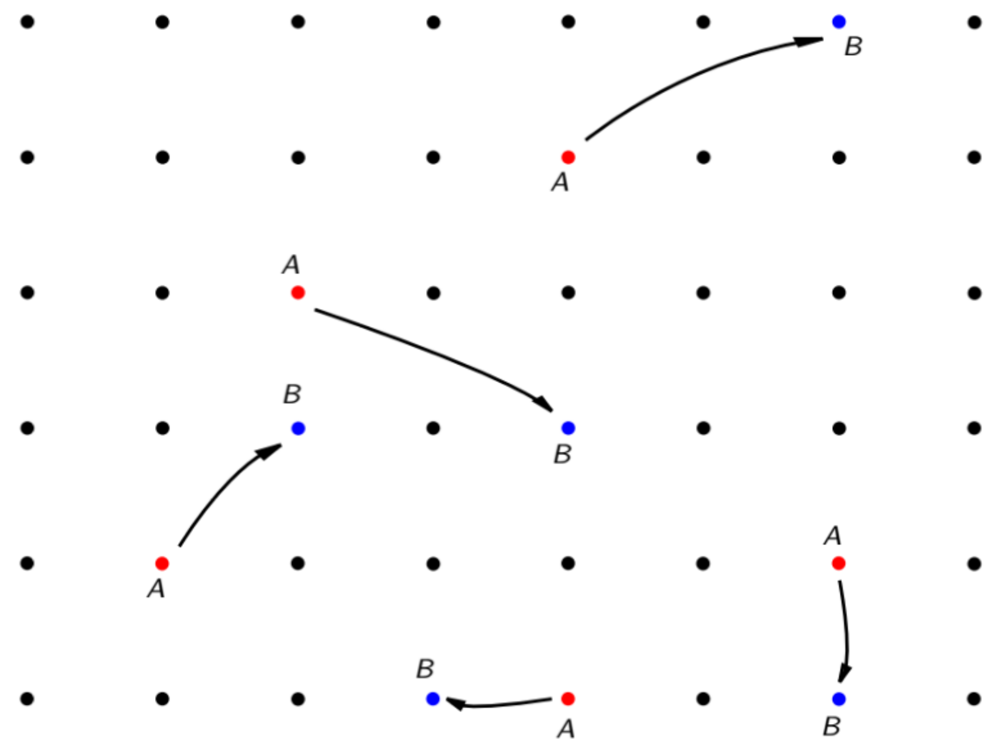




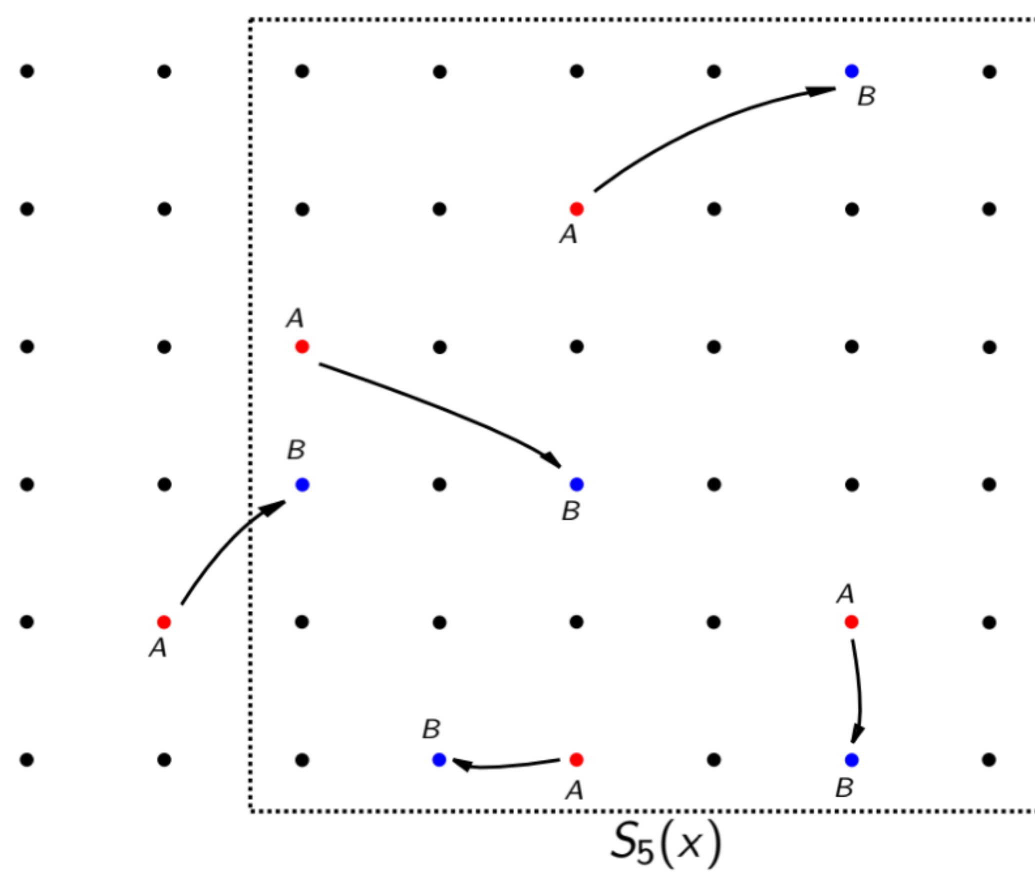
Let G be the graph with vertex set $\mathbb{T}^k$ where $x, y \in \mathbb{T}^k$ are adjacent if there is $g \in \mathbb{Z}^d$ such that $g \cdot x = y$ where $|g|_\infty = 1$.

To show A and B are equidecomposable, it suffices to find a Borel bijection $f\colon A \to B$ of bounded distance in G. (For some fixed N, for all $x \in A$, $d_G(x, f(x)) \leq N$).

Then if $A_g = \{x : f(x) = g \cdot x\}$, the sets $\{A_g\}_{|g|_\infty \leq N}$ partition A, and the sets $\{g \cdot A_g\}_{|g|_\infty \leq N}$ will partition B.



A picture of an equidecomposition viewed inside a single orbit of the action.



For an equidecomposition to exist, any sufficiently large "square" $S_N(x) = \{(n_1, \ldots, n_d) \cdot x \in \mathbb{Z}^d : 0 \leq n_i < N\}$ in the orbit must contain roughly the same number of elements of A and B.

By the ergodic theorem, we would expect $|S_N(x) \cap A| \approx \lambda(A)N^d$.

遍历理论。似乎是基础但完全不会，已经看不下去了。

The key to Laczkovich's proof is a strong quantitative refinement of the ergodic theorem for translation actions, using ideas from Diophantine approximation and discrepancy theory.

Lemma (Laczkovich 1992 after Schmidt, Niederreiter-Wills)

For A, B and the action as above, $\exists\epsilon > 0$ and M such that for every x and N,

$$\left| S_N(x) \cap A - \lambda(A)N^d \right| \le MN^{d-1-\epsilon}$$

and

$$\left| S_N(x) \cap B - \lambda(B)N^d \right| \le MN^{d-1-\epsilon}$$

Roughly, every square $S_N(x)$ contains very close to $\lambda(A)N^d$ many elements of A and B.

很显然由于基础的过度缺失，已经不能继续了。

需补充：① 张量代数 ② 群作用 ③ 代数拓扑 (π^k) ④ 遍历论 ⑤ 测度论

到此为止了。最后放上论文作者构造的集合：