Basic Topology

§6.1 Boundedness

Prototypical example for this section: $[0, 1]$ *is bounded but* $\mathbb{R}$ *is not.*

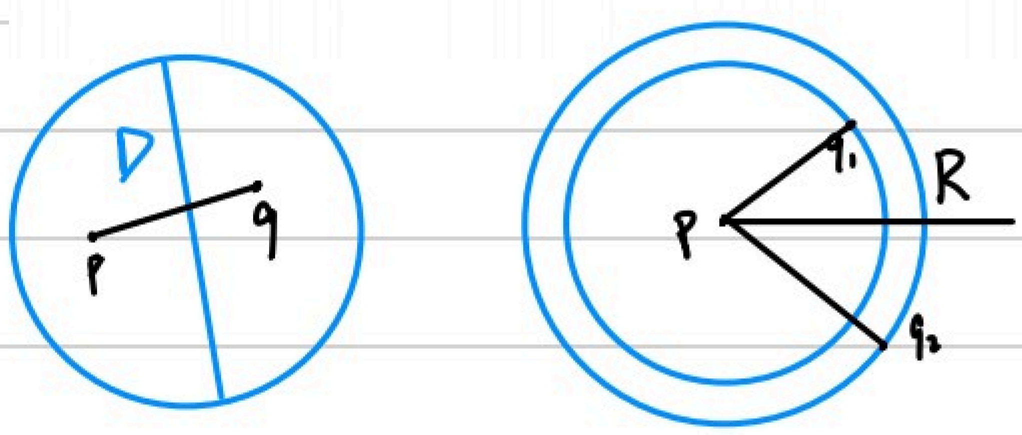
Here is one notion of how to prevent a metric space from being a bit too large.

Definition 6.1.1. A metric space M is **bounded** if there is a constant D such that $d(p, q) \le D$ for all $p, q \in M$.

You can change the order of the quantifiers:

Proposition 6.1.2 (Boundedness with radii instead of diameters)
A metric space M is bounded if and only if for every point $p \in M$, there is a radius R (possibly depending on p) such that $d(p, q) \le R$ for all $q \in M$.

Exercise 6.1.3. Use the triangle inequality to show these are equivalent. (The names "radius" and "diameter" are a big hint!)



（直观地看）

Example 6.1.4 (Examples of bounded spaces)

(a) Finite intervals like $[0, 1]$ and (a, b) are bounded.

(b) The unit square $[0, 1]^2$ is bounded.

(c) $\mathbb{R}^n$ is not bounded for any $n \ge 1$.

(d) A discrete space on an infinite set is bounded.

(e) $\mathbb{N}$ is not bounded, despite being homeomorphic to the discrete space!

唔，看了看剩下的内容，觉得还是回顾一下之前学的那比较好。

In Analysis, we studied real numbers a lot. We defined many properties such as convergence of sequences and continuity of functions. For example, if (x_n) is a sequence in $\mathbb{R}$, $x_n \to x$ means

$$(\forall \varepsilon > 0)(\exists N)(\forall n > N)\,|x_n - x| < \varepsilon.$$

Similarly, a function f is continuous at x_0 if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\,|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Given a set X, we can define a *metric* ("distance function") $d : X \times X \to \mathbb{R}$, where $d(x, y)$ is the distance between the points x and y. Then we say a sequence (x_n) in X converges to x if

$$(\forall \varepsilon > 0)(\exists N)(\forall n > N)\, d(x_n, x) < \varepsilon.$$

Similarly, a function $f : X \to X$ is continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\, d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon.$$

Of course, we will need the metric d to satisfy certain conditions such as being non-negative, and we will explore these technical details in the first part of the course.

1 Metric spaces

1.1 Definitions

As mentioned in the introduction, given a set X, it is often helpful to have a notion of distance between points. This distance function is known as the *metric*.

Definition (Metric space)**.** A *metric space* is a pair (X, d_X) where X is a set (the *space*) and d_X is a function $d_X : X \times X \to \mathbb{R}$ (the *metric*) such that for all x, y, z,

- $d_X(x, y) \geq 0$ (non-negativity)
- $d_X(x, y) = 0$ iff $x = y$ (identity of indiscernibles)
- $d_X(x, y) = d_X(y, x)$ (symmetry)
- $d_X(x, z) \leq d_X(x, y) + d_X(y, z)$ (triangle inequality)

We will have two quick examples of metrics before going into other important definitions. We will come to more examples afterwards.

Definition (Metric subspace)**.** Let (X, d_X) be a metric space, and $Y \subseteq X$. Then (Y, d_Y) is a metric space, where $d_Y(a, b) = d_X(a, b)$, and said to be a *subspace* of X.

Definition (Convergent sequences)**.** Let (x_n) be a sequence in a metric space (X, d_X). We say (x_n) *converges to* $x \in X$, written $x_n \to x$, if $d(x_n, x) \to 0$ (as a real sequence). Equivalently,

$$(\forall \varepsilon > 0)(\exists N)(\forall n > N)\, d(x_n, x) < \varepsilon.$$

Example.

- Let $(\mathbf{v}_n)$ be a sequence in $\mathbb{R}^k$ with the Euclidean metric. Write $\mathbf{v}_n = (v_n^1, \cdots, v_n^k)$, and $\mathbf{v} = (v^1, \cdots, v^k) \in \mathbb{R}^k$. Then $\mathbf{v}_n \to \mathbf{v}$ iff $(v_n^i) \to v^i$ for all i.

– Let X have the discrete metric, and suppose $x_n \to x$. Pick $\varepsilon = \frac{1}{2}$. Then there is some N such that $d(x_n, x) < \frac{1}{2}$ whenever $n > N$. But if $d(x_n, x) < \frac{1}{2}$, we must have $d(x_n, x) = 0$. So $x_n = x$. Hence if $x_n \to x$, then eventually all x_n are equal to x.

Proposition. If (X, d) is a metric space, (x_n) is a sequence in X such that $x_n \to x$, $x_n \to x'$, then $x = x'$.

证明：这个证明的确很巧妙啊。(唔，就那样)

$\because \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $d(x_n, x) < \varepsilon/2$ if $n > N$

and $\exists N' \in \mathbb{N}$, s.t. $d(x_n, x') < \varepsilon/2$ if $n > N'$

$\therefore$ if $n > \max(N, N')$, we have

$0 \le d(x, x') \le d(x, x_n) + d(x_n, x') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\because \forall \varepsilon > 0$, $0 \le d(x, x') < \varepsilon$

$\therefore d(x, x') = 0 \quad \therefore x = x'$

Definition (Continuous function). Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$. We say f is *continuous* if $f(x_n) \to f(x)$ (in Y) whenever $x_n \to x$ (in X).

Example. Let $X = \mathbb{R}$ with the Euclidean metric. Let $Y = \mathbb{R}$ with the discrete metric. Then $f : X \to Y$ that maps $f(x) = x$ is not continuous. This is since $1/n \to 0$ in the Euclidean metric, but not in the discrete metric.

On the other hand, $g : Y \to X$ by $g(x) = x$ is continuous, since a sequence in Y that converges is eventually constant.

下面是一些例子。虽然每本书都会给，但这里给的格外有趣。

Example (Manhattan metric). Let $X = \mathbb{R}^2$, and define the metric as

$$d(\mathbf{x}, \mathbf{y}) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

The first three axioms are again trivial. To prove the triangle inequality, we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &\ge |x_1 - z_1| + |x_2 - z_2| \\ &= d(\mathbf{x}, \mathbf{z}), \end{aligned}$$

using the triangle inequality for $\mathbb{R}$.

Example (British railway metric). Let $X = \mathbb{R}^2$. We define

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} |\mathbf{x} - \mathbf{y}| & \text{if } \mathbf{x} = k\mathbf{y} \\ |\mathbf{x}| + |\mathbf{y}| & \text{otherwise} \end{cases}$$

To explain the name of this metric, think of Britain with London as the origin. Since the railway system is ~~stupid~~ less than ideal, all trains go through London.

For example, if you want to go from Oxford to Cambridge (and obviously not the other way round), you first go from Oxford to London, then London to Cambridge. So the distance traveled is the distance from London to Oxford plus the distance from London to Cambridge.

The exception is when the two destinations lie along the same line, in which case, you can directly take the train from one to the other without going through London, and hence the "if $\mathbf{x} = k\mathbf{y}$" clause.

Example (p-adic metric)**.** Let $p \in \mathbb{Z}$ be a prime number. We first define the norm $|n|_p$ to be p^{-k}, where k is the highest power of p that divides n. If $n = 0$, we let $|n|_p = 0$. For example, $|20|_2 = |2^2 \cdot 5|_2 = 2^{-2}$.

Now take $X = \mathbb{Z}$, and let $d_p(a, b) = |a - b|_p$. The first three axioms are trivial, and the triangle inequality can be proved by making some number-theoretical arguments about divisibility.

This metric has rather interesting properties. With respect to d_2, we have $1, 2, 4, 8, 16, 32, \cdots \to 0$, while $1, 2, 3, 4, \cdots$ does not converge. We can also use it to prove certain number-theoretical results, but we will not go into details here.

Example (Uniform metric)**.** Let $X = C[0, 1]$ be the set of all continuous functions on $[0, 1]$. Then define

$$d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

The maximum always exists since continuous functions on $[0, 1]$ are bounded and attain their bounds.

Now let $F : C[0, 1] \to \mathbb{R}$ be defined by $F(f) = f(\frac{1}{2})$. Then this is continuous with respect to the uniform metric on $C[0, 1]$ and the usual metric on $\mathbb{R}$:

Let $f_n \to f$ in the uniform metric. Then we have to show that $F(f_n) \to F(f)$, i.e. $f_n(\frac{1}{2}) \to f(\frac{1}{2})$. This is easy, since we have

$$0 \le |F(f_n) - F(f)| = |f_n(\tfrac{1}{2}) - f(\tfrac{1}{2})| \le \max |f_n(x) - f(x)| \to 0.$$

So $|f_n(\frac{1}{2}) - f(\frac{1}{2})| \to 0$. So $f_n(\frac{1}{2}) \to f(\frac{1}{2})$.

Definition (Norm)**.** Let V be a real vector space. A *norm* on V is a function $\| \cdot \| : V \to \mathbb{R}$ such that 范数（在欧氏空间就是向量长度。）

- $\|\mathbf{v}\| \ge 0$ for all $\mathbf{v} \in V$
- $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- $\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$
- $\|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

Example. Let $V = \mathbb{R}^n$. There are several possible norms we can define on $\mathbb{R}^n$:

$$\|\mathbf{v}\|_1 = \sum_{i=1}^n |v_i|$$

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^n v_i^2}$$

$$\|\mathbf{v}\|_\infty = \max\{|v_i| : 1 \leq i \leq n\}.$$

In general, we can define the norm

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p}.$$

for any $1 \leq p \leq \infty$, and $\|\mathbf{v}\|_\infty$ is the limit as $p \to \infty$.

Proof that these are indeed norms is left as an exercise for the reader (in the example sheets).

Lemma. If $\|\cdot\|$ is a norm on V, then

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

defines a metric on V.

Proof.

(i) $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| \geq 0$ by the definition of the norm.

(ii) $d(\mathbf{v}, \mathbf{w}) = 0 \Leftrightarrow \|\mathbf{v} - \mathbf{w}\| = 0 \Leftrightarrow \mathbf{v} - \mathbf{w} = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{w}$.

(iii) $d(\mathbf{w}, \mathbf{v}) = \|\mathbf{w} - \mathbf{v}\| = \|(-1)(\mathbf{v} - \mathbf{w})\| = |-1|\|\mathbf{v} - \mathbf{w}\| = d(\mathbf{v}, \mathbf{w})$.

(iv) $d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}) = \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| \geq \|\mathbf{u} - \mathbf{w}\| = d(\mathbf{u}, \mathbf{w})$. □

We'll now define the inner product of a real vector space. This is a generalization of the notion of the "dot product".

Definition (Inner product)**.** Let V be a real vector space. An *inner product* on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that

(i) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in V$

(ii) $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

(iii) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.

(iv) $\langle \mathbf{v}_1 + \lambda \mathbf{v}_2, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle + \lambda \langle \mathbf{v}_2, \mathbf{w} \rangle$.

(i) Let $V = \mathbb{R}^n$. Then

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v_i w_i$$

is an inner product.

(ii) Let $V = C[0,1]$. Then

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$$

is an inner product.

We just showed that norms induce metrics. The proof was completely trivial as the definitions were almost the same. Now we want to prove that inner products induce norms. However, this is slightly less trivial. To do so, we need the Cauchy-Schwarz inequality. ([illegible] "induce"?)

Theorem (Cauchy-Schwarz inequality). If $\langle \cdot, \cdot \rangle$ is an inner product, then

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle.$$

Proof. For any x, we have

$$\langle \mathbf{v} + x\mathbf{w}, \mathbf{v} + x\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle + 2x\langle \mathbf{v}, \mathbf{w} \rangle + x^2\langle \mathbf{w}, \mathbf{w} \rangle \geq 0.$$

Seen as a quadratic in x, since it is always non-negative, it can have at most one real root. So

好家伙，这比 conway 上的证明妙

$$(2\langle \mathbf{v}, \mathbf{w} \rangle)^2 - 4\langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle \leq 0.$$

So the result follows. □

Lemma. If $\langle \cdot, \cdot \rangle$ is an inner product on V, then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

is a norm.

Proof.

(i) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \geq 0$.

(ii) $\|\mathbf{v}\| = 0 \Leftrightarrow \langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$.

(iii) $\|\lambda \mathbf{v}\| = \sqrt{\langle \lambda \mathbf{v}, \lambda \mathbf{v} \rangle} = \sqrt{\lambda^2 \langle \mathbf{v}, \mathbf{v} \rangle} = |\lambda| \|\mathbf{v}\|$.

(iv)

$$\begin{aligned}
(\|\mathbf{v}\| + \|\mathbf{w}\|)^2 &= \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 \\
&\geq \langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \\
&= \|\mathbf{v} + \mathbf{w}\|^2
\end{aligned}$$

□

Definition (Open and closed balls). Let (X, d) be a metric space. For any $x \in X$, $r \in \mathbb{R}$,

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

is the *open ball* centered at x.

$$\bar{B}_r(x) = \{y \in X : d(y, x) \leq r\}$$

is the *closed ball* centered at x.

Definition (Open subset). $U \subseteq X$ is an *open subset* if for every $x \in U$, $\exists \delta > 0$ such that $B_\delta(x) \subseteq U$.

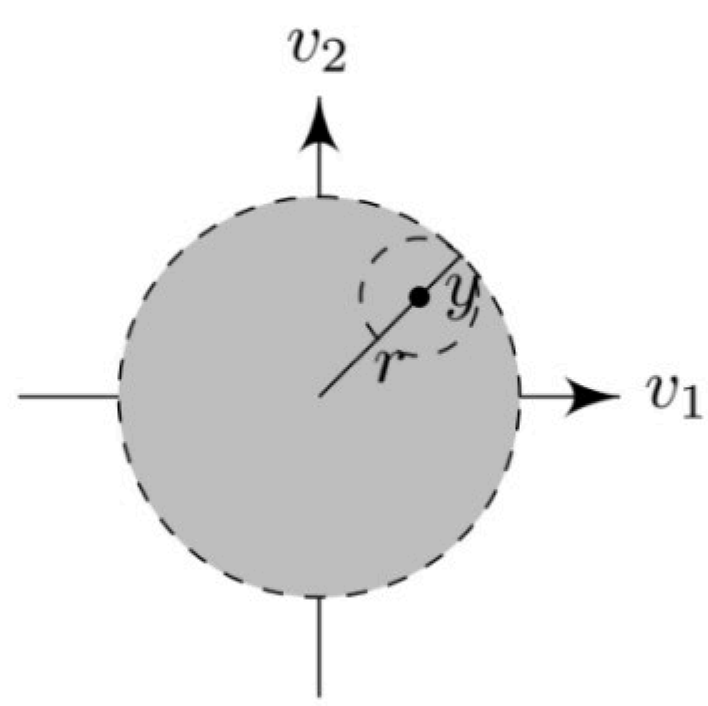
$C \subseteq X$ is a *closed subset* if $X \setminus C \subseteq X$ is open.

As if we've not emphasized this enough, this is a very *very* important definition.

We first prove that this is a sensible definition.

Lemma. The open ball $B_r(x) \subseteq X$ is an open subset, and the closed ball $\bar{B}_r(x) \subseteq X$ is a closed subset.

Proof. Given $y \in B_r(x)$, we must find $\delta > 0$ with $B_\delta(y) \subseteq B_r(x)$.



Since $y \in B_r(x)$, we must have $a = d(y, x) < r$. Let $\delta = r - a > 0$. Then if $z \in B_\delta(y)$, then

$$d(z, x) \le d(z, y) + d(y, x) < (r - a) + a = r.$$

So $z \in B_r(x)$. So $B_\delta(y) \subseteq B_r(x)$ as desired.

The second statement is equivalent to $X \setminus \bar{B}_r(x) = \{y \in X : d(y, x) > r\}$ is open. The proof is very similar. $\square$

Definition (Open neighborhood). If $x \in X$, an *open neighborhood* of x is an open $U \subseteq X$ with $x \in U$.

Definition (Limit point). Let $A \subseteq X$. Then $x \in X$ is a *limit point* of A if there is a sequence $x_n \to x$ such that $x_n \in A$ for all n.

Intuitively, a limit point is a point we can get arbitrarily close to.

(i) If $a \in A$, then a is a limit point of A, by taking the sequence $a, a, a, a, \cdots$.

(ii) If $A = (0, 1) \subseteq \mathbb{R}$, then 0 is a limit point of A, e.g. take $x_n = \frac{1}{n}$.

(iii) Every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$. And this is how we define $\mathbb{R}$ (Dedekind)

Proposition. $C \subseteq X$ is a closed subset if and only if every limit point of C is an element of C.

（挺符合直觉的。在 metric space 里 closed set 都"带边界"，反观这些边界点对于 open set 来说也是 limit point 但并不 $\in$ 自身。(~~这么说在metric space里开是开闭是闭~~) 不像 topology space 里还有 clopen）

证明：($\Rightarrow$) Suppose C is closed, $x_n \to x$ while $x_n \in C$, we need to show that $x \in C$.

$\because$ C closed $\therefore$ $X \setminus C$ open

$\therefore$ if $x \notin C$, then $x \in X\setminus C$, which means $X \setminus C$ is an open neighbourhood of x.

According to lemma, (I forgot to paste)

Lemma. If U is an open neighbourhood of x and $x_n \to x$, then $\exists N$ such that $x_n \in U$ for all $n > N$.

Proof. Since U is open, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq U$. Since $x_n \to x$, $\exists N$ such that $d(x_n, x) < \delta$ for all $n > N$. This implies that $x_n \in B_\delta(x)$ for all $n > N$. So $x_n \in U$ for all $n > N$. □

there exists N s.t. $x_n \in X\setminus C$ when $n > N$, which make a contradiction.

($\Leftarrow$) we can prove the inverse negative proposition.

Suppose C is not closed, we need to show there exists a limit point not in C.

C is not closed $\Leftrightarrow$ $X \setminus C$ is not open

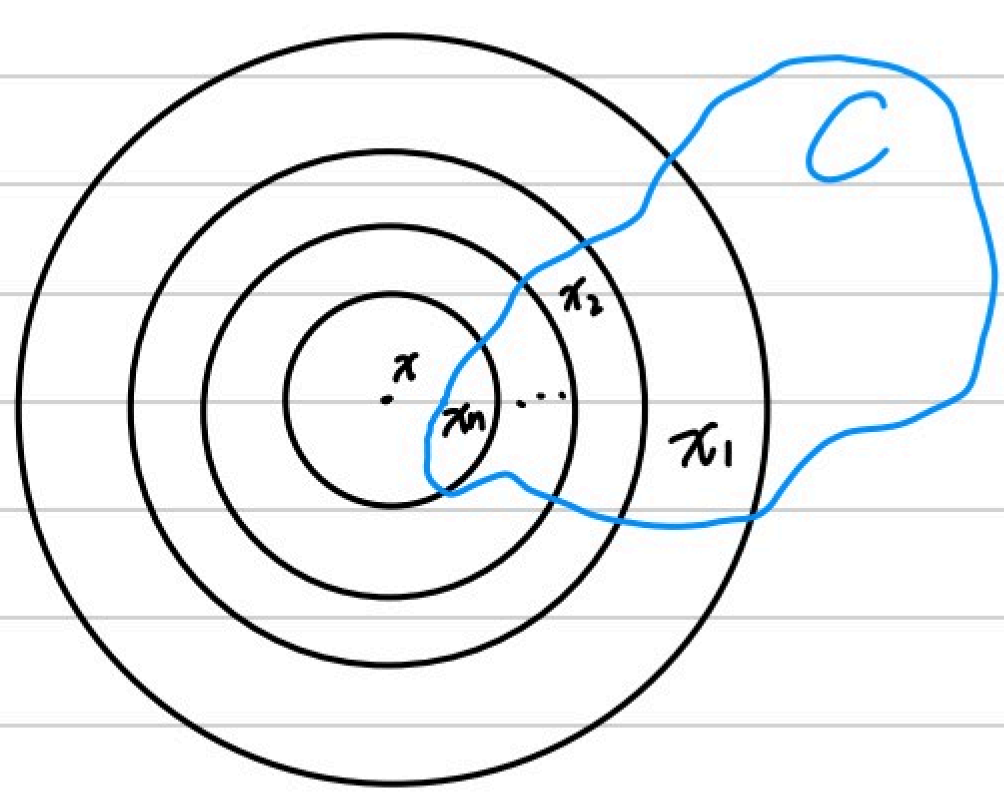
$\Leftrightarrow \exists x \in X\setminus C, \forall \delta > 0, B_\delta(x) \nsubseteq A$

$\Leftrightarrow \forall \delta > 0, B_\delta(x) \cap C \neq \varnothing$

$\therefore$ pick $x_n \in B_{\frac{1}{n}}(x) \cap C$ for each n, then $d(x_n, x) = \frac{1}{n} \to 0$

$\therefore$ $x_n \to x$, x is a limit point but $x \in X \setminus C$ □

直观地画一下($\Leftarrow$)部分：



Proposition (Characterization of continuity)**.** Let (X, d_x) and (Y, d_y) be metric spaces, and $f: X \to Y$. The following conditions are equivalent:

(i) f is continuous

(ii) If $x_n \to x$, then $f(x_n) \to f(x)$ (which is the definition of continuity)

(iii) For any closed subset $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

(iv) For any open subset $U \subseteq Y$, $f^{-1}(U)$ is open in X.

(v) For any $x \in X$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Alternatively, $d_x(x, z) < \delta \Rightarrow d_y(f(x), f(z)) < \varepsilon$.

The third and fourth condition can allow us to immediately decide if a subset is open or closed in some cases.

Example. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as

$$f(x_1, x_2, x_3) = x_1^2 + x_2^4 x_3^6 + x_1^8 x_3^2.$$

Then this is continuous. So $\{\mathbf{x} \in \mathbb{R}^3 : f(\mathbf{x}) \leq 1\} = f^{-1}((-\infty, 1])$ is closed in $\mathbb{R}^3$.

Lemma.

(i) $\emptyset$ and X are open subsets of X.

(ii) Suppose $V_\alpha \subseteq X$ is open for all $\alpha \in A$. Then $U = \bigcup_{\alpha \in A} V_\alpha$ is open in X. 无限并

(iii) If $V_1, \cdots, V_n \subseteq X$ are open, then so is $V = \bigcap_{i=1}^{n} V_i$. 有限交

我们由此即可进入拓扑空间（下面换 Munkres）

定义 集合 X 上的一个**拓扑**(topology)乃是 X 的子集的一个族 $\mathcal{T}$，它满足以下条件：

(1) $\varnothing$ 和 X 在 $\mathcal{T}$ 中.

(2) $\mathcal{T}$ 的任意子族的元素的并在 $\mathcal{T}$ 中.

(3) $\mathcal{T}$ 的任意有限子族的元素的交在 $\mathcal{T}$ 中.

一个指定了拓扑 $\mathcal{T}$ 的集合 X 叫做一个**拓扑空间**(topological space).

确切地说，一个拓扑空间就是一个有序偶对 $(X, \mathcal{T})$，其中 X 是一个集合，$\mathcal{T}$ 是 X 上的一个拓扑. 在不致混淆的情况下，常常不专门提到 $\mathcal{T}$.

例 2 如果 X 为任意的一个集合，X 所有子集的族是 X 的一个拓扑，称之为**离散拓扑**(discrete topology). 仅由 X 和 $\varnothing$ 组成的族也是 X 的一个拓扑，称之为**密着拓扑**(indiscrete topology)或**平庸拓扑**(trivial topology). ■

图 12.2

例 3 设 X 是一个集合，$\mathcal{T}_f$ 是使得 $X-U$ 或者是有限集或者等于 X 的那些 X 的子集 U 的全体. 那么 $\mathcal{T}_f$ 是 X 上的一个拓扑，称之为**有限补拓扑**(finite complement topology). 因为 $X-X=\varnothing$ 是有限集，$X-\varnothing=X$，所以 X 与 $\varnothing$ 都在 $\mathcal{T}_f$ 中. 若 $\{U_\alpha\}$ 是 $\mathcal{T}_f$ 中非空元素的一个加标族，为了证明 $\bigcup U_\alpha$ 在 $\mathcal{T}_f$ 中，只要确定

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha).$$

因为每一个集合 $X-U_\alpha$ 是有限集，所以右边的集合是一个有限集. 如果 $U_1, \cdots, U_n$ 是 $\mathcal{T}_f$ 的非空元素，为了证明 $\bigcap U_i$ 在 $\mathcal{T}_f$ 中，只要确定

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i).$$

因为上式右边是有限集的有限并，所以是有限集. ■

例 4 设 X 是一个集合. $\mathcal{T}_c$ 是使得 $X-U$ 是可数集或者等于 X 的所有 X 的子集 U 的全体. 容易验证，$\mathcal{T}_c$ 是 X 上的一个拓扑. ■

证明：$X - \varnothing = X \in \mathcal{T}_c$ $X - X = \varnothing$. 可数 $\in \mathcal{T}_c$

对于集族$\{U_\alpha\}$，为确定$\bigcup U_\alpha \in \mathcal{T}_c$，只需验证$X-\bigcup U_\alpha$可数

$\because X-\bigcup U_\alpha = \bigcap (X-U_\alpha)$，$X-U_\alpha$均可数

$\therefore X-\bigcup U_\alpha$可数 $\therefore \bigcup U_\alpha \in \mathcal{T}_c$.

为确定$\bigcap U_i \in \mathcal{T}_c$，$i=1,2,\cdots,n$，只需验证$X-\bigcap U_i$可数

$\because X-\bigcap U_i = \bigcup (X-U_i)$，为有限个可数集的并

$\therefore X-\bigcap U_i$可数 $\therefore \bigcap U_i \in \mathcal{T}_c$

$\therefore \mathcal{T}_c$是一个拓扑

定义 设$\mathcal{T}$和$\mathcal{T}'$是给定集合X上的两个拓扑，如果$\mathcal{T}' \supset \mathcal{T}$，则称$\mathcal{T}'$**细于**(finer)$\mathcal{T}$. 若$\mathcal{T}' \supset \mathcal{T}$是真包含关系，则称$\mathcal{T}'$**严格细于**(strictly finer)$\mathcal{T}$. 这两种情形有时也分别称之为$\mathcal{T}$**粗于**(coarser)$\mathcal{T}'$，和$\mathcal{T}$**严格粗于**(strictly coarser)$\mathcal{T}'$. 我们说$\mathcal{T}$与$\mathcal{T}'$是可比较的，如果或者$\mathcal{T}' \supset \mathcal{T}$或者$\mathcal{T} \supset \mathcal{T}'$.

受这个术语的启发，我们不妨把拓扑空间设想成一辆装满石子的卡车，一堆石子以及若干堆石子合在一起都是开集. 现在把石子敲成小碎块，那么开集族就变大了. 因此，按照这种运算，拓扑，如同石子一样，就变细了.

定义 如果X是一个集合. X的某拓扑的一个**基**(basis)是X的子集的一个族$\mathcal{B}$(其成员称为**基元素**(basis element))，满足条件：

(1)对于每一个$x \in X$，至少存在一个包含x的基元素B.

(2)若x属于两个基元素B_1和B_2的交，则存在包含x的一个基元素B_3，使得$B_3 \subset B_1 \cap B_2$.

我之前学拓扑似乎就是到这开始迷惑的。

如果$\mathcal{B}$满足以上两个条件，我们定义**由$\mathcal{B}$生成的拓扑**$\mathcal{T}$(topology $\mathcal{T}$ generated by $\mathcal{B}$)如下：如果对于每一个$x \in U$，存在一个基元素$B \in \mathcal{B}$，使得$x \in B$并且$B \subset U$，那么X的子集U称为X的开集(即是$\mathcal{T}$的一个元素).

例 1 设$\mathcal{B}$为平面上所有圆形域(圆周的内部)所组成的族，则$\mathcal{B}$满足基的定义中的两个条件. 第二个条件如图 13.1 所示. 在由$\mathcal{B}$生成的拓扑中，平面的一个子集U是开集，是指对于U中任意x，都有含于U中的某一个圆域. ■

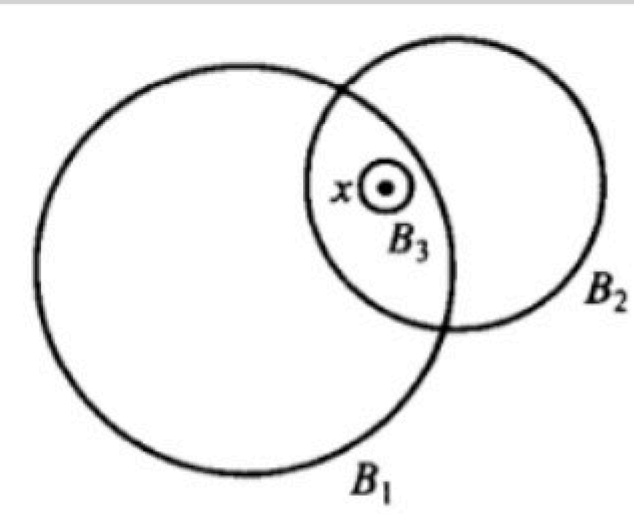
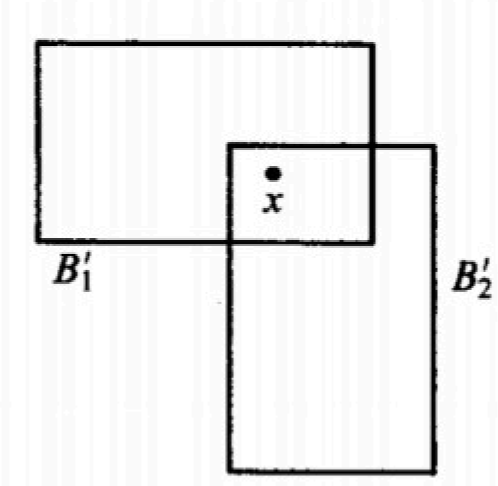


图 13.1

例 2 设$\mathcal{B}'$为平面上所有矩形域(矩形的内部)的族，其中矩形的边平行于两个坐标轴，则$\mathcal{B}'$满足基的定义中的两个条件. 第二个条件如图 13.2 所示. 这个条件是显然满足的，这是由于任何两个基元素的交，其本身就是一个基元素(或者是空集). 以后我们将会看到，在平面上由基$\mathcal{B}'$生成的拓扑与例 1 中基$\mathcal{B}$生成的拓扑是一样的. ■



例 3 设X是任意的一个集合. X的所有单点子集的族是X上离散拓扑的一个基. ■

引理 13.1 设X是一个集合，$\mathcal{B}$是X的拓扑$\mathcal{T}$的一个基. 则$\mathcal{T}$等于$\mathcal{B}$中元素所有并的族①.

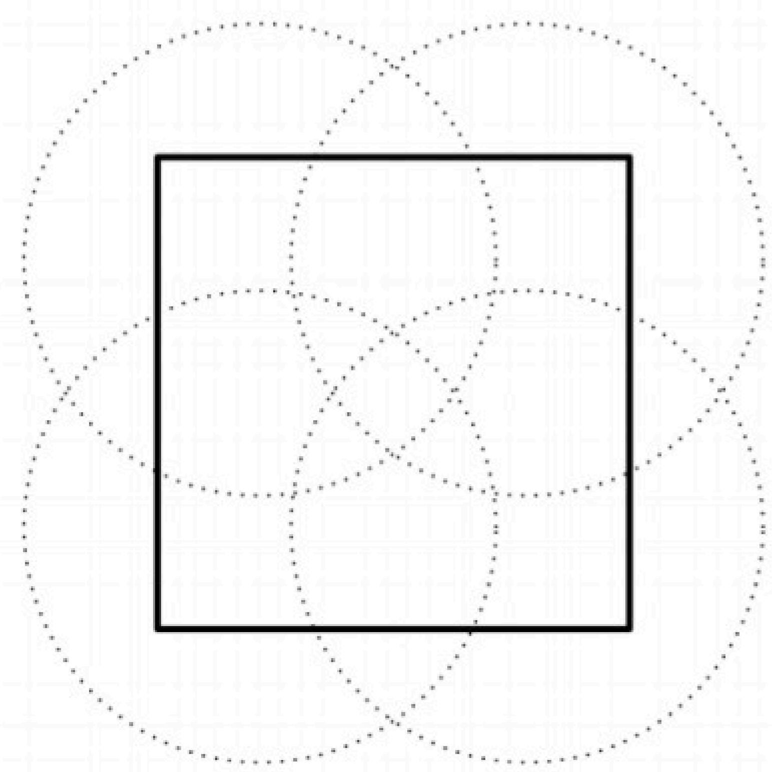
证 给定$\mathcal{B}$中元素的一个族，这些$\mathcal{B}$的元素也是$\mathcal{T}$的元素. 由于$\mathcal{T}$是一个拓扑，于是它们的并也在$\mathcal{T}$中. 反之，给定$U \in \mathcal{T}$，对于每一个$x \in U$，选取$\mathcal{B}$的一个元素B_x，使得$x \in B_x \subset U$. 从而，U等于$\mathcal{B}$中元素的一个并. ■

先看到这里。我们回到Napkin.（我记得之前有三个令我头疼的概念，似乎在 Lee 的 GTM 202 里）

继续看 metric space，回到 boundness：

Definition 6.1.5. A metric space is **totally bounded** if for any $\varepsilon > 0$, we can cover M with finitely many ε-neighborhoods.

For example, if $\varepsilon = 1/2$, you can cover $[0,1]^2$ by ε-neighborhoods.



Example 6.1.7 (Examples of totally bounded spaces)

(a) A subset of $\mathbb{R}^n$ is bounded if and only if it is totally bounded.

This is for Euclidean geometry reasons: for example in $\mathbb{R}^2$ if I can cover a set by a single disk of radius 2, then I can certainly cover it by finitely many disks of radius 1/2. (We won't prove this rigorously.)

(b) So for example $[0,1]$ or $[0,2] \times [0,3]$ is totally bounded.

(c) In contrast, a discrete space on an infinite set is not totally bounded.

§6.2 Completeness

Prototypical example for this section: *$\mathbb{R}$ is complete, but $\mathbb{Q}$ and $(0,1)$ are not.*

So far we can only talk about sequences converging if they have a limit. But consider the sequence

$$x_1 = 1,\ x_2 = 1.4,\ x_3 = 1.41,\ x_4 = 1.414, \dots.$$

It converges to $\sqrt{2}$ in $\mathbb{R}$, of course. But it fails to converge in $\mathbb{Q}$; there is no *rational* number this sequence converges to. And so somehow, if we didn't know about the existence of $\mathbb{R}$, we would have *no idea* that the sequence (x_n) is "approaching" something.

That seems to be a shame. Let's set up a new definition to describe these sequences whose terms **get close to each other**, even if they don't approach any particular point in the space. Thus, we only want to mention the given points in the definition.

Definition 6.2.1. Let $x_1, x_2, \dots$ be a sequence which lives in a metric space $M = (M, d_M)$. We say the sequence is **Cauchy** if for any $\varepsilon > 0$, we have

$$d_M(x_m, x_n) < \varepsilon$$

for all sufficiently large m and n.

Question 6.2.2. Show that a sequence which converges is automatically Cauchy. (Draw a picture.)

根据定义，$\exists N \in \mathbb{N}$，$\forall n > N$，$d(x, x_n) < \varepsilon/2$

$\therefore$ 令 $m = N+1$，$n = N+2$，则有

$0 \le d(x_n, x) < d(x_m, x) < \varepsilon/2$

$\therefore d(x_m, x_n) \le d(x_m, x) + d(x_n, x) < \varepsilon$

Definition 6.2.3. A metric space M is **complete** if every Cauchy sequence converges.

Example 6.2.4 (Examples of complete spaces)

(a) $\mathbb{R}$ is complete. (Depending on your definition of $\mathbb{R}$, this either follows by definition, or requires some work. We won't go through this here.)

(b) The discrete space is complete, as the only Cauchy sequences are eventually constant.

(c) The closed interval $[0, 1]$ is complete.

(d) $\mathbb{R}^n$ is complete as well. (You're welcome to prove this by induction on n.)

Example 6.2.5 (Non-examples of complete spaces)

(a) The rationals $\mathbb{Q}$ are not complete.

(b) The open interval $(0, 1)$ is not complete, as the sequence 0.9, 0.99, 0.999, 0.9999, ... is Cauchy but does not converge.

累了，先到这里。