Informal Notes on Mathematics

2022.12.21

阳性第二天。继续来看拓扑。上次到完备性了。

So, metric spaces need not be complete, like $\mathbb{Q}$. But we certainly would like them to be complete, and in light of the following theorem this is not unreasonable.

Theorem 6.2.6 (Completion)
Every metric space can be "completed", i.e. made into a complete space by adding in some points.

We won't need this construction at all, so it's left as Problem $6C^\dagger$.

Example 6.2.7 ($\mathbb{Q}$ completes to $\mathbb{R}$)
The completion of $\mathbb{Q}$ is $\mathbb{R}$.

(In fact, by using a modified definition of completion not depending on the real numbers, other authors often use this as the definition of $\mathbb{R}$.)

§6.3 Let the buyer beware

There is something suspicious about both these notions: neither are preserved under homeomorphism!

Example 6.3.1 (Something fishy is going on here)
Let $M = (0,1)$ and $N = \mathbb{R}$. As we saw much earlier M and N are homeomorphic. However:

- $(0,1)$ is totally bounded, but not complete.
- $\mathbb{R}$ is complete, but not bounded.

This is the first hint of something going awry with the metric. As we progress further into our study of topology, we will see that in fact *open and closed sets* (which we motivated by using the metric) are the notion that will really shine later on. I insist on introducing the metric first so that the standard pictures of open and closed sets make sense, but eventually it becomes time to remove the training wheels.

(我才意识到他在前面还讲过一段拓扑)

Definition 2.4.1. Let M and N be metric spaces. A function $f\colon M \to N$ is a **homeomorphism** if it is a bijection, and both $f\colon M \to N$ and its inverse $f^{-1}\colon N \to M$ are continuous. We say M and N are **homeomorphic**.

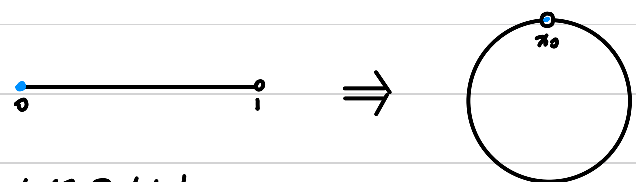
Needless to say, homeomorphism is an equivalence relation.

You might be surprised that we require f^{-1} to also be continuous. Here's the reason: you can show that if ϕ is an isomorphism of groups, then ϕ^{-1} also preserves the group operation, hence ϕ^{-1} is itself an isomorphism. The same is not true for continuous bijections, which is why we need the new condition.

Example 2.4.2 (Homeomorphism $\neq$ continuous bijection)

(a) There is a continuous bijection from $[0, 1)$ to the circle, but it has no continuous inverse.

(b) Let M be a discrete space with size $|\mathbb{R}|$. Then there is a continuous function $M \to \mathbb{R}$ which certainly has no continuous inverse.



由左及右很显然连续，但对于 x_0 点，由右向左时，方向会 $\to 0$，方向会 $\to 1$，不连续

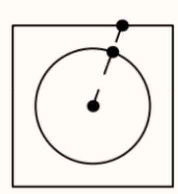
Note that this is the topologist's definition of "same" – homeomorphisms are "continuous deformations". Here are some examples.

Example 2.4.3 (Examples of homeomorphisms)

(a) Any space M is homeomorphic to itself through the identity map.

(b) The old saying: a doughnut (torus) is homeomorphic to a coffee cup. (Look this up if you haven't heard of it.)

(c) The unit circle S^1 is homeomorphic to the boundary of the unit square. Here's one bijection between them, after an appropriate scaling:



Example 2.4.5 (Homeomorphisms really don't preserve size)

Surprisingly, the open interval $(-1, 1)$ is homeomorphic to the real line $\mathbb{R}$! One bijection is given by

$$x \mapsto \tan(x\pi/2)$$

with the inverse being given by $t \mapsto \frac{2}{\pi} \arctan(t)$.

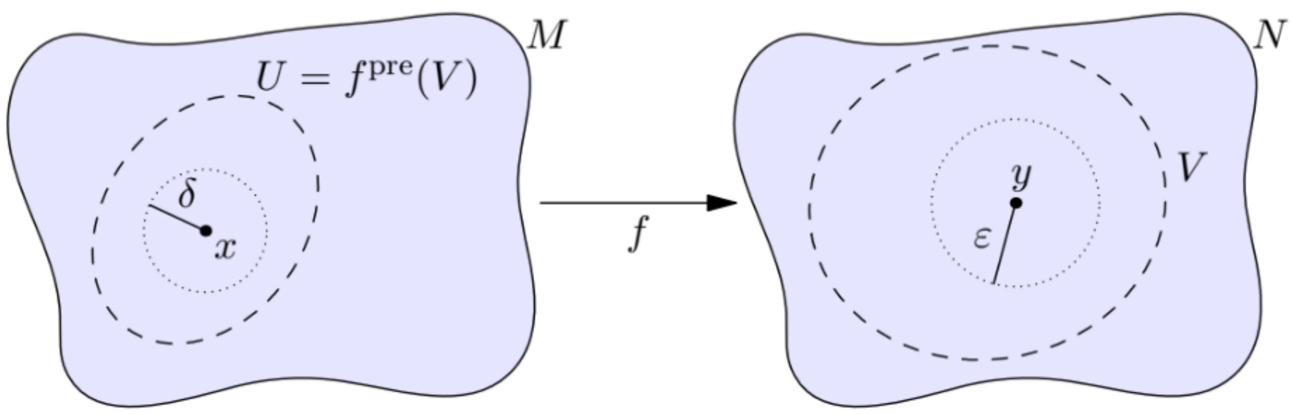
This might come as a surprise, since $(-1, 1)$ doesn't look that much like $\mathbb{R}$; the former is "bounded" while the latter is "unbounded".

补充一下一个定理的证明（之前应该也提到过）

Theorem 2.6.11 (Open set condition)

A function $f\colon M \to N$ of metric spaces is continuous if and only if the pre-image of every open set in N is open in M.

Now assume f is continuous. First, suppose V is an open subset of the metric space N; let $U = f^{\text{pre}}(V)$. Pick $x \in U$, so $y = f(x) \in V$; we want an open neighborhood of x inside U.



As V is open, there is some small ε-neighborhood around y which is contained inside V. By continuity of f, we can find a δ such that the δ-neighborhood of x gets mapped by f into the ε-neighborhood in N, which in particular lives inside V. Thus the δ-neighborhood lives in U, as desired. $\square$

回到一开始的部分。我要纠正上次笔记的一个错误：metric space里集合可以clopen

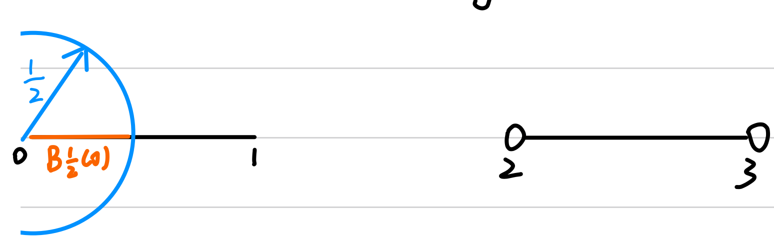
> **Exercise 6.4.3.** Let $M = [0,1] \cup (2,3)$. Show that $[0,1]$ and $(2,3)$ are both open and closed in M.

证明：We only need to show $[0,1]$ and $(2,3)$ are both open, which can show that they are also both closed automatically.

It's easy to show $(2,3)$ and $(0,1)$ are open since they're open balls in $\mathbb{R}$.

Then all we need to do is to show point 0 and 1 satisfy the rule.

It may not make sense, but notice that we're considering in M, and $B_{\frac{1}{2}}(0)$ in M look like this:



And it turns out that the sets are open in M. $\square$