§7.2 Re-definitions

Now that we've defined a topological space, for nearly all of our metric notions we can write down as the definition the one that required only open sets (which will of course agree with our old definitions when we have a metric space).

§7.2.i Continuity

Here was our motivating example, continuity:

Definition 7.2.1. We say function $f\colon X \to Y$ of topological spaces is **continuous** at a point $p \in X$ if the pre-image of any open neighborhood of $f(p)$ is an open neighborhood of p. The function is continuous if it is continuous at every point.

Thus homeomorphisms carries over: a bijection which is continuous in both directions.

Definition 7.2.2. A **homeomorphism** of topological spaces (X, τ_X) and (Y, τ_Y) is a bijection $f\colon X \to Y$ which induces a bijection from τ_X to τ_Y: i.e. the bijection preserves open sets.

Definition 7.2.4. In a general topological space X, we say that $S \subseteq X$ is **closed** in X if the complement $X \setminus S$ is open in X.

If $S \subseteq X$ is any set, the **closure** of S, denoted $\overline{S}$, is defined as the smallest closed set containing S.

Not everything works:

Remark 7.2.7 (Complete and (totally) bounded are metric properties) — The two metric properties we have seen, "complete" and "(totally) bounded", are not topological properties. They rely on a metric, so as written we cannot apply them to topological spaces. One might hope that maybe, there is some alternate definition (like we saw for "continuous function") that is just open-set based. But Example 6.3.1 showing $(0,1) \cong \mathbb{R}$ tells us that it is hopeless.

Remark 7.2.8 (Sequences don't work well) — You could also try to port over the notion of sequences and convergent sequences. However, this turns out to break a lot of desirable properties. Therefore I won't bother to do so, and thus if we are discussing sequences you should assume that we are working with a metric space.

As you might have guessed, there exist topological spaces which cannot be realized as metric spaces (in other words, are not **metrizable**). One example is just to take $X = \{a, b, c\}$ and the topology $\tau_X = \{\varnothing, \{a, b, c\}\}$. This topology is fairly "stupid": it can't tell apart any of the points a, b, c! But any metric space can tell its points apart (because $d(x, y) > 0$ when $x \neq y$).

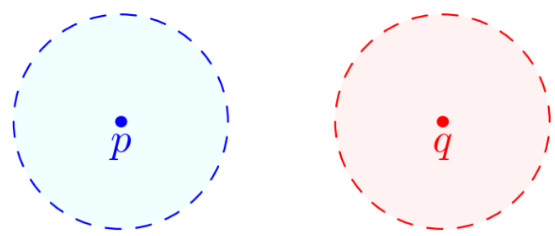
We'll see less trivial examples later, but for now we want to add a little more sanity condition onto our spaces. There is a whole hierarchy of such axioms, labelled T_n for integers n (with $n = 0$ being the weakest and $n = 6$ the strongest); these axioms are called **separation axioms**.

By far the most common hypothesis is the T_2 axiom, which bears a special name.

Definition 7.3.1. A topological space X is **Hausdorff** if for any two distinct points p and q in X, there exists an open neighborhood U of p and an open neighborhood V of q such that

$$U \cap V = \varnothing.$$

In other words, around any two distinct points we should be able to draw disjoint open neighborhoods. Here's a picture to go with above, but not much going on.



Question 7.3.2. Show that all metric spaces are Hausdorff.

I just want to define this here so that I can use this word later. In any case, basically any space we will encounter other than the Zariski topology is Hausdorff.

Q. 7.3.2. (The question isn't very rigorous. Say "topology induced by metric space is better.)

证明：(我以为我想得太简单了没想到就那么简单。)

Let $D = d(p,q)$ and $B_{D/2}(p)$ & $B_{D/2}(q)$ satisfy. $\square$

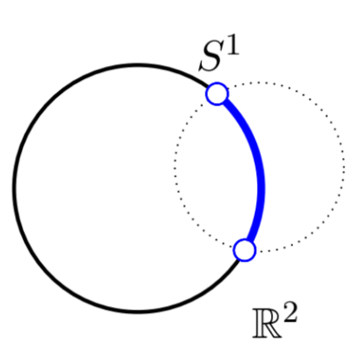
§7.4 Subspaces

Prototypical example for this section: S^1 is a subspace of $\mathbb{R}^2$.

One can also take subspaces of general topological spaces.

Definition 7.4.1. Given a topological space X, and a subset $S \subseteq X$, we can make S into a topological space by declaring that the open subsets of S are $U \cap S$ for open $U \subseteq X$. This is called the **subspace topology**.

So for example, if we view S^1 as a subspace of $\mathbb{R}^2$, then any open arc is an open set, because you can view it as the intersection of an open disk with S^1.



§7.5 Connected spaces

Prototypical example for this section: $[0,1] \cup [2,3]$ *is disconnected.*

Even in metric spaces, it is possible for a set to be both open and closed.

Definition 7.5.1. A subset S of a topological space X is **clopen** if it is both closed and open in X. (Equivalently, both S and its complement are open.)

For example $\varnothing$ and the entire space are examples of clopen sets. In fact, the presence of a nontrivial clopen set other than these two leads to a so-called *disconnected* space.

> **Question 7.5.2.** Show that a space X has a nontrivial clopen set (one other than $\varnothing$ and X) if and only if X can be written as a disjoint union of two nonempty open sets.

证明：($\Leftarrow$) Suppose $X = A \sqcup B$, $A \cdot B \neq \varnothing$ and both open. Then $A \cdot B$ is both clopen.

($\Rightarrow$) Suppose A is that nontrivial clopen set. The $X - A$ is open and $A \cap (X-A) = \varnothing$ $\square$

We say X is **disconnected** if there are nontrivial clopen sets, and **connected** otherwise. To see why this should be a reasonable definition, it might help to solve Problem 7A†.

Example 7.5.3 (Disconnected and connected spaces)

(a) The metric space

$$\{(x,y) \mid x^2+y^2 \le 1\} \cup \{(x,y) \mid (x-4)^2+y^2 \le 1\} \subseteq \mathbb{R}^2$$

is disconnected (it consists of two disks).

(b) The space $[0,1] \cup [2,3]$ is disconnected: it consists of two segments, each of which is a clopen set.

(c) A discrete space on more than one point is disconnected, since *every* set is clopen in the discrete space.

(d) Convince yourself that the set

$$\{x \in \mathbb{Q} : x^2 < 2014\}$$

is a clopen subset of $\mathbb{Q}$. Hence $\mathbb{Q}$ is disconnected too – it has *gaps*.

(e) $[0,1]$ is connected.

§7.6 Path-connected spaces

Prototypical example for this section: *Walking around in* $\mathbb{C}$.

A stronger and perhaps more intuitive notion of a connected space is a *path-connected* space. The short description: "walk around in the space".

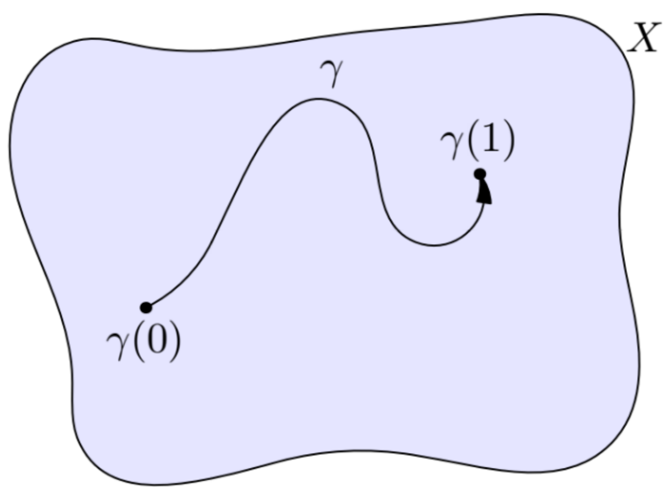
Definition 7.6.1. A **path** in the space X is a continuous function

$$\gamma : [0,1] \to X.$$

Its **endpoints** are the two points $\gamma(0)$ and $\gamma(1)$.

You can think of $[0,1]$ as measuring "time", and so we'll often write $\gamma(t)$ for $t \in [0,1]$ (with t standing for "time"). Here's a picture of a path.

好，马上就可以接上HoTT了。(唔，看来得明天了。)



Definition 7.6.3. A space X is **path-connected** if any two points in it are connected by some path.

> **Exercise 7.6.4** (Path-connected implies connected)**.** Let $X = U \sqcup V$ be a disconnected space. Show that there is no path from a point of U to point V. (If $\gamma\colon [0,1] \to X$, then we get $[0,1] = \gamma^{\text{pre}}(U) \sqcup \gamma^{\text{pre}}(V)$, but $[0,1]$ is connected.)

Example 7.6.5 (Examples of path-connected spaces)

- $\mathbb{R}^2$ is path-connected, since we can "connect" any two points with a straight line.
- The unit circle S^1 is path-connected, since we can just draw the major or minor arc to connect two points.