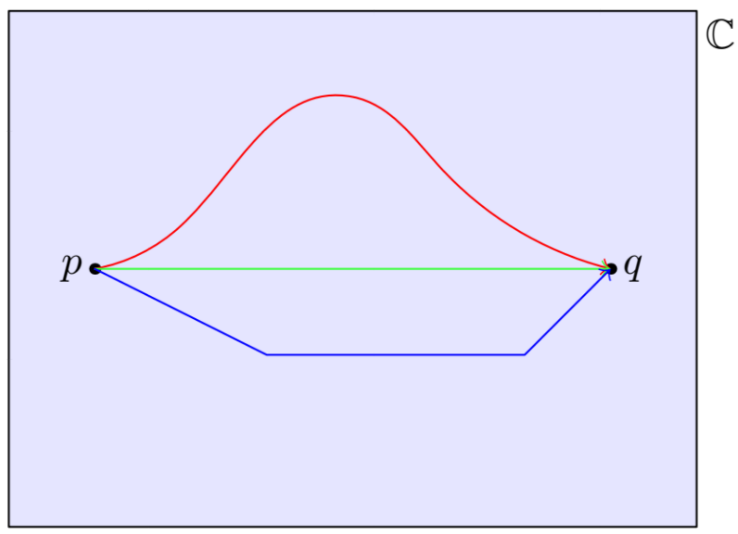
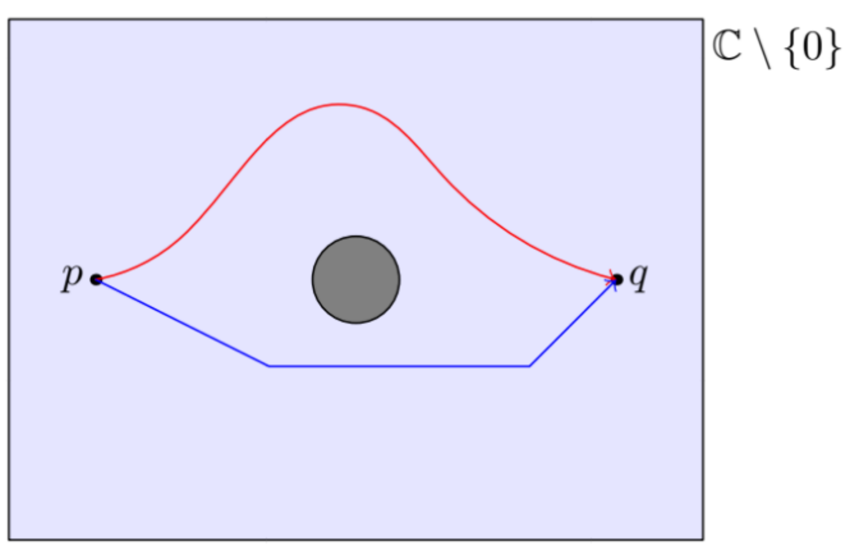
§7.7 Homotopy and simply connected spaces

Prototypical example for this section: $\mathbb{C}$ *and* $\mathbb{C} \setminus \{0\}$.

Now let's motivate the idea of homotopy. Consider the example of the complex plane $\mathbb{C}$ (which you can think of just as $\mathbb{R}^2$) with two points p and q. There's a whole bunch of paths from p to q but somehow they're not very different from one another. If I told you "walk from p to q" you wouldn't have too many questions.



So we're living happily in $\mathbb{C}$ until a meteor strikes the origin, blowing it out of existence. Then suddenly to get from p to q, people might tell you two different things: "go left around the meteor" or "go right around the meteor".



So what's happening? In the first picture, the red, green, and blue paths somehow all looked the same: if you imagine them as pieces of elastic string pinned down at p and q, you can stretch each one to any other one.

But in the second picture, you can't move the red string to match with the blue string: there's a meteor in the way. The paths are actually different.[3]

The formal notion we'll use to capture this is *homotopy equivalence*. We want to write a definition such that in the first picture, the three paths are all *homotopic*, but the two paths in the second picture are somehow not homotopic. And the idea is just continuous deformation.

Definition 7.7.1. Let α and β be paths in X whose endpoints coincide. A (path) **homotopy** from α to β is a continuous function $F : [0,1]^2 \to X$, which we'll write $F_s(t)$ for $s, t \in [0,1]$, such that

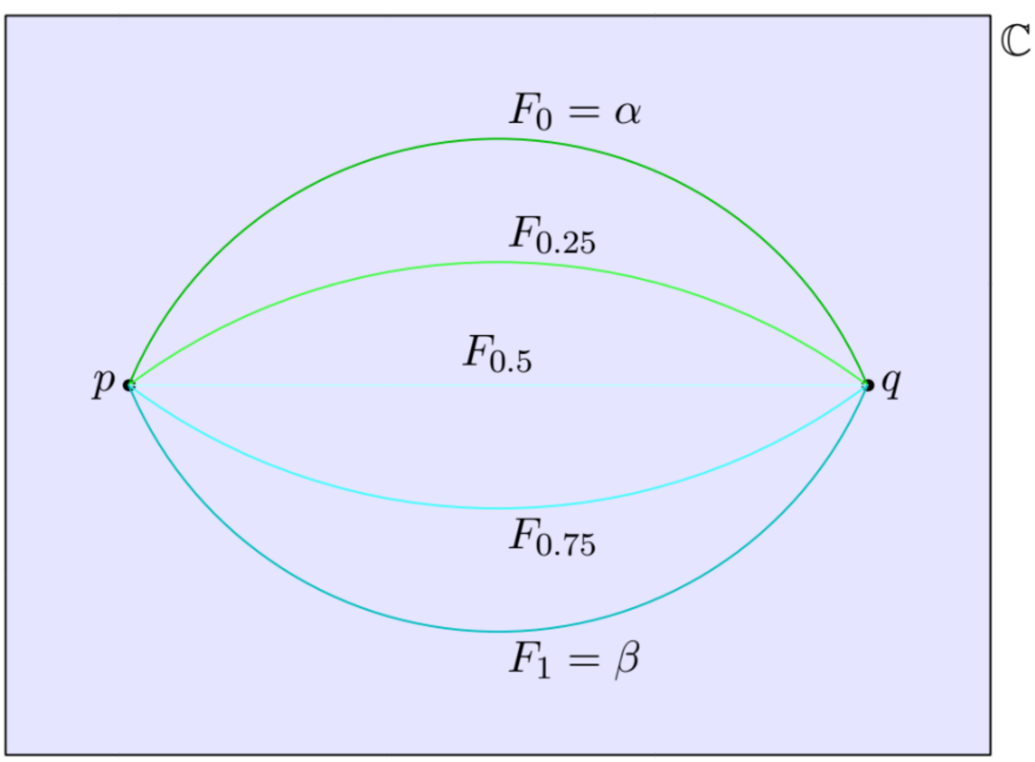
$$F_0(t) = \alpha(t) \text{ and } F_1(t) = \beta(t) \text{ for all } t \in [0,1]$$

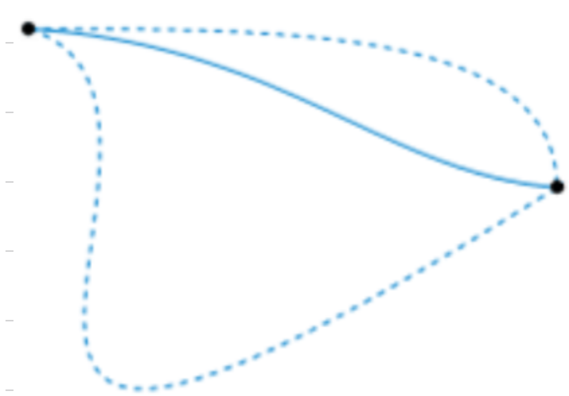
and moreover

$$\alpha(0) = \beta(0) = F_s(0) \text{ and } \alpha(1) = \beta(1) = F_s(1) \text{ for all } s \in [0,1].$$

If a path homotopy exists, we say α and β are path **homotopic** and write $\alpha \simeq \beta$.

What this definition is doing is taking α and "continuously deforming" it to β, while keeping the endpoints fixed. Note that for each particular s, F_s is itself a function. So s represents time as we deform α to β: it goes from 0 to 1, starting at α and ending at β.





So now I can tell you what makes $\mathbb{C}$ special:

Definition 7.7.4. A space X is **simply connected** if it's path-connected and for any points p and q, all paths from p to q are homotopic.

That's why you don't ask questions when walking from p to q in $\mathbb{C}$: there's really only one way to walk. Hence the term "simply" connected.

阳性第四天，难受了一早上，到下午4:00才舒服一点，可以看点别的了。

Definition 1.1. *A* **semigroup** *is a nonempty set* **G** *together with a binary operation on* **G** *which is*

(i) *associative:* a(bc) = (ab)c *for all* a, b, c ε **G**;

a **monoid** *is a semigroup* G *which contains a*

(ii) (*two-sided*) *identity element* $e \in G$ *such that* $ae = ea = a$ *for all* $a \in G$.

A **group** *is a monoid* G *such that*

(iii) *for every* $a \in G$ *there exists a* (*two-sided*) *inverse element* $a^{-1} \in G$ *such that* $a^{-1}a = aa^{-1} = e$.

A semigroup G *is said to be* **abelian** *or* **commutative** *if its binary operation is*

(iv) *commutative:* $ab = ba$ *for all* $a,b \in G$.

Our principal interest is in groups. However, semigroups and monoids are convenient for stating certain theorems in the greatest generality. Examples are given below. The **order** of a group G is the cardinal number $|G|$. G is said to be finite [resp. infinite] if $|G|$ is finite [resp. infinite].

时隔两年来看 Hungerford，怀旧。

Proposition 1.3. *Let* G *be a semigroup. Then* G *is a group if and only if the following conditions hold:*

(i) *there exists an element* $e \in G$ *such that* $ea = a$ *for all* $a \in G$ (*left identity element*);
(ii) *for each* $a \in G$, *there exists an element* $a^{-1} \in G$ *such that* $a^{-1}a = e$ (*left inverse*).

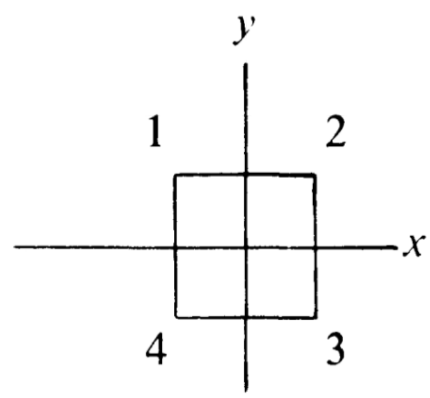
Proof: (⇒) Trivial. (⇐) We have a left identity element. Now we need to show it's two-sided.
$(a \cdot a^{-1})(a \cdot a^{-1}) = a \cdot (a^{-1}a)a^{-1} = aea^{-1} = aa^{-1}$ $\therefore aa^{-1} = e$ $\therefore a^{-1}$ is also the right inverse.
$ae = a(a^{-1}a) = (a \cdot a^{-1})a = ea = a$ $\therefore e$ is two-sided. $\therefore G$ is a group $\square$

Proposition 1.4. *Let* G *be a semigroup. Then* G *is a group if and only if for all* $a, b \in G$ *the equations* $ax = b$ *and* $ya = b$ *have solutions in* G.

Proof: (⇒) Trivial. (⇐) $a^{-1} \cdot ax = a^{-1}b \Rightarrow x = a^{-1}b$, $\therefore a \cdot a^{-1}b = b$
$\therefore a \cdot a^{-1} = e$ $\therefore a^{-1}$ is two-sided inverse.
$a \cdot e = a(a^{-1}a) = (a \cdot a^{-1})a = ea = a$ $\therefore e$ is two-sided. $\therefore G$ is a group. $\square$

EXAMPLES. The integers **Z**, the rational numbers **Q**, and the real numbers **R** are each infinite abelian groups under ordinary addition. Each is a monoid under ordinary multiplication, but not a group (0 has no inverse). However, the nonzero elements of **Q** and **R** respectively form infinite abelian groups under multiplication. The even integers under multiplication form a semigroup that is not a monoid.

EXAMPLE. Consider the square with vertices consecutively numbered 1,2,3,4, center at the origin of the *x*-*y* plane, and sides parallel to the axes.



Let $D_4{}^*$ be the following set of "transformations" of the square. $D_4{}^* = \{R,R^2,R^3,I,T_x,T_y,T_{1,3},T_{2,4}\}$, where R is a counterclockwise rotation about the center of 90°, R^2 a counterclockwise rotation of 180°, R^3 a counterclockwise rotation of 270°

EXAMPLE. Let S be a nonempty set and $A(S)$ the set of all bijections $S \to S$. Under the operation of composition of functions, $f \circ g$, $A(S)$ is a group, since composition is associative, composition of bijections is a bijection, 1_S is a bijection, and every bijection has an inverse (see (13) of Introduction, Section 3). The elements of $A(S)$ are called **permutations** and $A(S)$ is called the group of permutations on the set S. If $S = \{1,2,3,\ldots,n\}$, then $A(S)$ is called the **symmetric group on n letters** and denoted S_n. Verify that $|S_n| = n!$ (Exercise 5). The groups S_n play an important role in the theory of finite groups.

Since an element σ of S_n is a function on the finite set $S = \{1,2,\ldots,n\}$, it can be described by listing the elements of S on a line and the image of each element under σ directly below it: $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & & i_n \end{pmatrix}$. The product $\sigma\tau$ of two elements of S_n is the composition function τ *followed by* σ; that is, the function on S given by $k \mapsto \sigma(\tau(k))$.[1] For instance, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ be elements of S_4. Then under $\sigma\tau$, $1 \mapsto \sigma(\tau(1)) = \sigma(4) = 4$, etc.; thus $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$; similarly, $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$. This example also shows that S_n need not be abelian.

Theorem 1.5. *Let* R $(\sim)$ *be an equivalence relation on a monoid* G *such that* $a_1 \sim a_2$ *and* $b_1 \sim b_2$ *imply* $a_1b_1 \sim a_2b_2$ *for all* $a_i, b_i \in G$. *Then the set* G/R *of all equivalence classes of* G *under* R *is a monoid under the binary operation defined by* $(\bar{a})(\bar{b}) = \overline{ab}$, *where* $\bar{x}$ *denotes the equivalence class of* $x \in G$. *If* G *is an* [*abelian*] *group, then so is* G/R.

An equivalence relation on a monoid G that satisfies the hypothesis of the theorem is called a **congruence relation** on G.

2. HOMOMORPHISMS AND SUBGROUPS

Essential to the study of any class of algebraic objects are the functions that preserve the given algebraic structure in the following sense.

Definition 2.1. *Let* G *and* H *be semigroups. A function* $f : G \to H$ *is a* **homomorphism** *provided*

$$f(ab) = f(a)f(b) \quad \textit{for all} \quad a,b \in G.$$

If f *is injective as a map of sets,* f *is said to be a* **monomorphism.** *If* f *is surjective,* f *is called an* **epimorphism.** *If* f *is bijective,* f *is called an* **isomorphism.** *In this case* G *and* H *are said to be* **isomorphic** (*written* $G \cong H$). *A homomorphism* $f : G \to G$ *is called an* **endomorphism** *of* G *and an isomorphism* $f : G \to G$ *is called an* **automorphism** *of* G.

If $f : G \to H$ and $g : H \to K$ are homomorphisms of semigroups, it is easy to see that $gf : G \to K$ is also a homomorphism. Likewise the composition of monomorphisms is a monomorphism; similarly for epimorphisms, isomorphisms and automorphisms. If G and H are groups with identities e_G and e_H respectively and $f : G \to H$ is a homomorphism, then $f(e_G) = e_H$; however, this is not true for monoids (Exercise 1). Furthermore $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$ (Exercise 1).

EXAMPLE. The map $f : \mathbf{Z} \to Z_m$ given by $x \mapsto \bar{x}$ (that is, each integer is mapped onto its equivalence class in Z_m) is an epimorphism of additive groups. f is called the canonical epimorphism of $\mathbf{Z}$ onto Z_m. Similarly, the map $g : \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ given by $r \mapsto \bar{r}$ is also an epimorphism of additive groups.

EXAMPLE. Given groups G and H, there are four homomorphisms: $G \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} G \times H \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} H$, given by $\iota_1(g) = (g,e)$; $\iota_2(h) = (e,h)$; $\pi_1(g,h) = g$; $\pi_2(g,h) = h$. ι_i is a monomorphism and π_j is an epimorphism $(i,j = 1,2)$.

Kerf、生成子群明天再说吧。