Category Theory (Continued)

1.2.4. *Example: abelian groups.* The abelian groups, along with group homomorphisms, form a category *Ab*.

1.2.5. *Important Example: Modules over a ring.* If A is a ring, then the A-modules form a category Mod_A. (This category has additional structure; it will be the prototypical example of an *abelian category*, see §1.6.) Taking $A = k$, we obtain Example 1.2.3; taking $A = \mathbb{Z}$, we obtain Example 1.2.4.

1.2.6. *Example: rings.* There is a category *Rings*, where the objects are rings, and the morphisms are maps of rings in the usual sense (maps of sets which respect addition and multiplication, and which send 1 to 1 by our conventions, §0.3).

1.2.7. *Example: topological spaces.* The topological spaces, along with continuous maps, form a category *Top*. The isomorphisms are homeomorphisms.

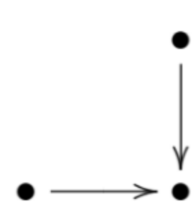
1.2.8. *Example: partially ordered sets.* A **partially ordered set**, (or **poset**), is a set S along with a binary relation $\geq$ on S satisfying:

(i) $x \geq x$ (reflexivity),
(ii) $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity), and
(iii) if $x \geq y$ and $y \geq x$ then $x = y$ (antisymmetry).

A partially ordered set $(S, \geq)$ can be interpreted as a category whose objects are the elements of S, and with a single morphism from x to y if and only if $x \geq y$ (and no morphism otherwise).

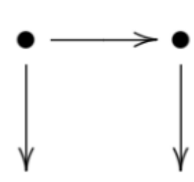
A trivial example is $(S, \geq)$ where $x \geq y$ if and only if $x = y$. Another example is

(1.2.8.1)



Here there are three objects. The identity morphisms are omitted for convenience, and the two non-identity morphisms are depicted. A third example is

(1.2.8.2)



Here the "obvious" morphisms are again omitted: the identity morphisms, and the morphism from the upper left to the lower right. Similarly,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

depicts a partially ordered set, where again, only the "generating morphisms" are depicted.

1.2.9. *Example: the category of subsets of a set, and the category of open subsets of a topological space.* If X is a set, then the subsets form a partially ordered set, where the order is given by inclusion. Informally, if $U \subset V$, then we have exactly one morphism $U \to V$ in the category (and otherwise none). Similarly, if X is a topological space, then the *open* sets form a partially ordered set, where the order is given by inclusion.

1.2.10. *Definition.* A **subcategory** $\mathscr{A}$ of a category $\mathscr{B}$ has as its objects some of the objects of $\mathscr{B}$, and some of the morphisms, such that the morphisms of $\mathscr{A}$ include the identity morphisms of the objects of $\mathscr{A}$, and are closed under composition. (For example, (1.2.8.1) is in an obvious way a subcategory of (1.2.8.2). Also, we have an obvious "inclusion functor" $i \colon \mathscr{A} \to \mathscr{B}$.)

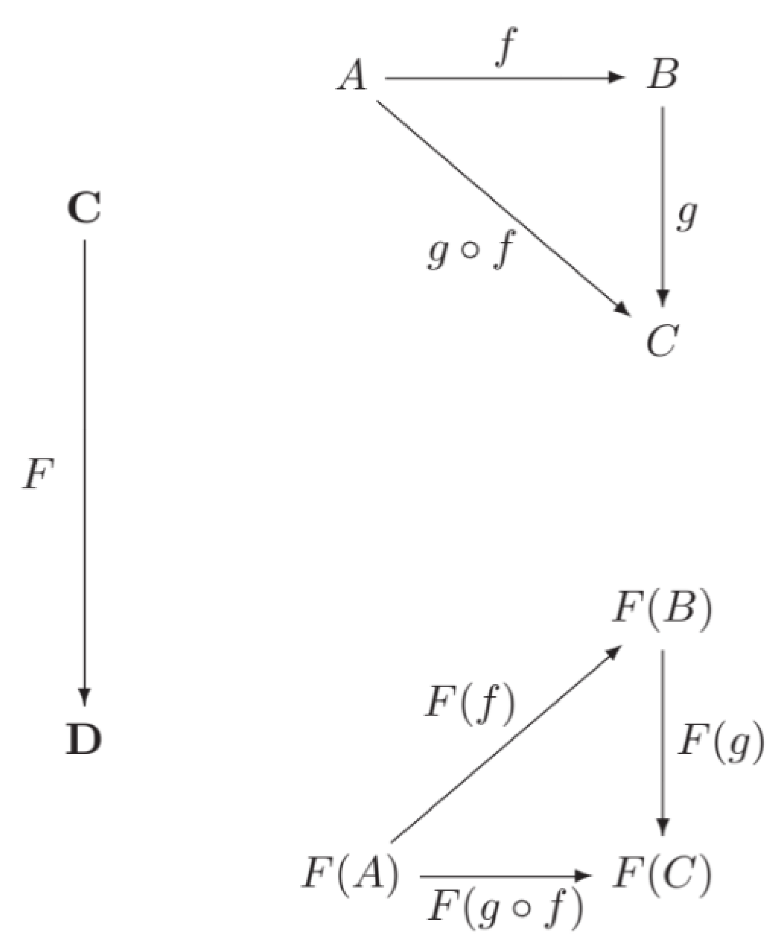
Definition 1.2. A *functor*

$$F : \mathbf{C} \to \mathbf{D}$$

between categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping of objects to objects and arrows to arrows, in such a way that

(a) $F(f : A \to B) = F(f) : F(A) \to F(B)$,
(b) $F(1_A) = 1_{F(A)}$,
(c) $F(g \circ f) = F(g) \circ F(f)$.

That is, F preserves domains and codomains, identity arrows, and compostion. A functor $F : \mathbf{C} \to \mathbf{D}$ thus gives a sort of "picture"—perhaps distorted—of $\mathbf{C}$ in $\mathbf{D}$.



1.2.12. *Example: a forgetful functor.* Consider the functor from the category of vector spaces (over a field k) Vec_k to *Sets*, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a **forgetful functor**, where some additional structure is forgotten. Another example of a forgetful functor is $Mod_A \to Ab$ from A-modules to abelian groups, remembering only the abelian group structure of the A-module.

1.2.13. *Topological examples.* Examples of covariant functors include the fundamental group functor π_1, which sends a topological space X with choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$ (what are the objects and morphisms of the source category?), and the ith homology functor $Top \to Ab$, which sends a topological space X to its ith homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a (continuous) morphism of pointed topological spaces $\phi: X \to Y$ with $\phi(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and similarly for homology groups. (同伦群什么的我看一遍忘一遍)

好，laugoyaauo，我们来(lyzuu61924)看一点代拓

§57.1 Spheres

Recall that

$$S^n = \{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$$

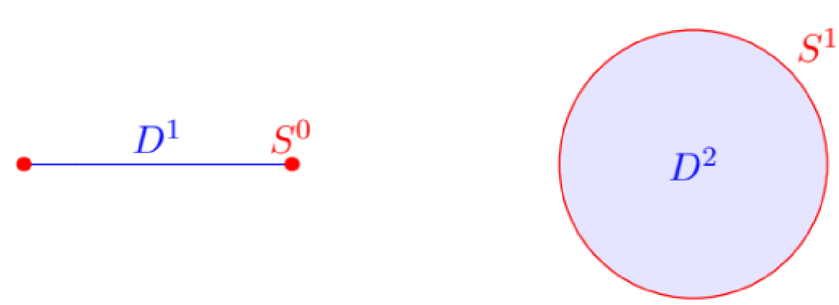
is the surface of an n-sphere while

$$D^{n+1} = \{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 \leq 1\} \subset \mathbb{R}^{n+1}$$

is the corresponding *closed ball* (So for example, D^2 is a disk in a plane while S^1 is the unit circle.)

> **Exercise 57.1.1.** Show that the open ball $D^n \setminus S^{n-1}$ is homeomorphic to $\mathbb{R}^n$.

In particular, S^0 consists of two points, while D^1 can be thought of as the interval $[-1, 1]$.



啊，稍等，我们点拓看得太少了，很多概念还没拓展到 Top Space 上，比如 homeomorphic.
考虑以下问题：

$$X = \{a, b, c, d, e\}, Y = \{g, h, i, j, k\},$$

$$\tau = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\},$$

和

$$\tau_1 = \{Y, \varnothing, \{g\}, \{i,j\}, \{g,i,j\}, \{h,i,j,k\}\}.$$

显然直觉上(X,τ)“等价于”(Y,τ_1). 由$f(a)=g, f(a)=g, f(a)=g, f(a)=g, f(a)=g$定义的映射$f:X\to Y$ 提供了等价性. 我们现在正式化这种等价性.

定义 4.2.1. *令(X,τ)和(Y,τ_1)为拓扑空间. 则它们称为***同胚***的, 如果存在满足下列性质的映射$f:X\to Y$:*

(i) *f是一一的(即由$f(x_1)=f(x_2)$有$x_1=x_2$成立),*

(ii) *f是到上的(即对任何$y\in Y$, 存在$x\in X$使得$f(x)=y$),*

(iii) *对每个$U\in\tau_1$, $f^{-1}(U)\in\tau$, 且*

(iv) *对每个$V\in\tau$, $f(V)\in\tau_1$.*

*进一步, 映射f称为(X,τ)和(Y,τ_1)之间的一个***同胚映射***. 我们记为$(X,\tau)\cong(Y,\tau_1)$.*

我们将要证出“$\cong$”是一个等价关系, 并用这一点证明所有开区间(a,b)彼此间是同胚的. 例4.2.2是第一步, 它表明了“$\cong$”是一个传递关系.

例 4.2.2. *令(X,τ), (Y,τ_1)和(Z,τ_2)为拓扑空间. 如果$(X,\tau)\cong(Y,\tau_1)$且$(Y,\tau_1)\cong(Z,\tau_2)$, 求证$(X,\tau)\cong(Z,\tau_2)$.*

证明.

由于$(X,\tau)\cong(Y,\tau_1)$且$(Y,\tau_1)\cong(Z,\tau_2)$, 存在同胚映射$f:(X,\tau)\to(Y,\tau_1)$和$g:(Y,\tau_1)\to(Z,\tau_2)$. 考虑符合映射$g\circ f:X\to Z$. [这里$g\circ f(x)=g(f(x))$, 对所有$x\in X$.] 验证$g\circ f$是一一的及到上的是程序性任务. 现在设$U\in\tau_2$. 则由于$g$为同胚映射, $g^{-1}(U)\in\tau_1$. 由f是同胚映射的事实, 我们得到$f^{-1}(g^{-1}(U))\in\tau$. 但$f^{-1}(g^{-1}(U))=(g\circ f)^{-1}(U)$. 所以$g\circ f$有定义4.2.1的性质(iii). 接下来令$V\in\tau$. 那么$f(V)\in\tau_1$, 故$g(f(V))\in\tau_2$; 即$g\circ f(V)\in\tau_2$, 于是我们看到$g\circ f$有定义4.2.1的性质(iv). 因此$g\circ f$是同胚映射. ∎

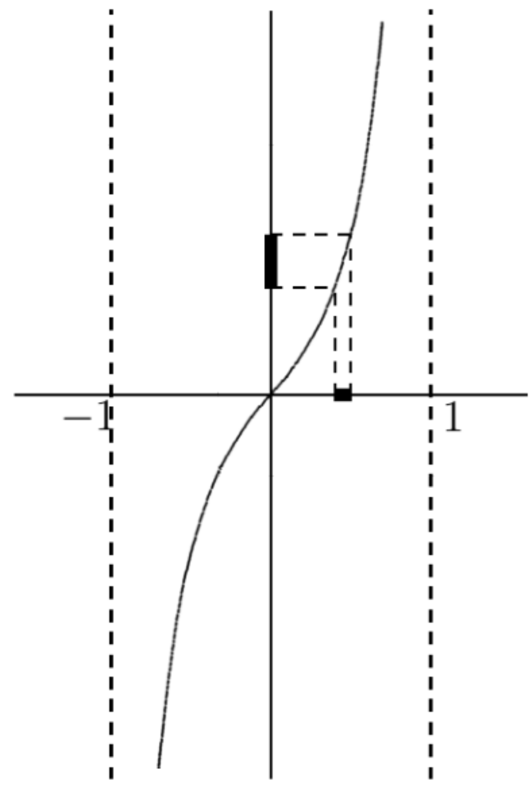
例 4.2.5. *求证空间$\mathbb{R}$与具有通常拓扑的开区间$(-1,1)$同胚*

证明.

(简略证明.) 定义$f:(-1,1)\to\mathbb{R}$,

$$f(x) = \frac{x}{1 - |x|}.$$

容易验证f是一一并到上的, 类似于例4.2.4中的图示推导表明f是一个同胚拓扑.



用同样的思路，令 $f: (x_1, x_2, \cdots, x_n) \to \left(\frac{x_1}{1-|x_1|}, \frac{x_2}{1-|x_2|}, \cdots, \frac{x_n}{1-|x_n|}\right)$，易知$f$为$D^n \setminus S^{n-1}$到$\mathbb{R}^n$的同胚

§57.2 Quotient topology

Prototypical example for this section: $D^n/S^{n-1} = S^n$, *or the torus.*

Given a space X, we can *identify* some of the points together by any equivalence relation $\sim$; for an $x \in X$ we denote its equivalence class by $[x]$. Geometrically, this is the space achieved by welding together points equivalent under $\sim$.

Formally,

Definition 57.2.1. Let X be a topological space, and $\sim$ an equivalence relation on the points of X. Then $X/\sim$ is the space whose

- Points are equivalence classes of X, and
- $U \subseteq X/\sim$ is open if and only if $\{x \in X \text{ such that } [x] \in U\}$ is open in X.

As far as I can tell, this definition is mostly useless for intuition, so here are some examples.

明天继续。