INFORMAL NOTES ON MATHEMATICS

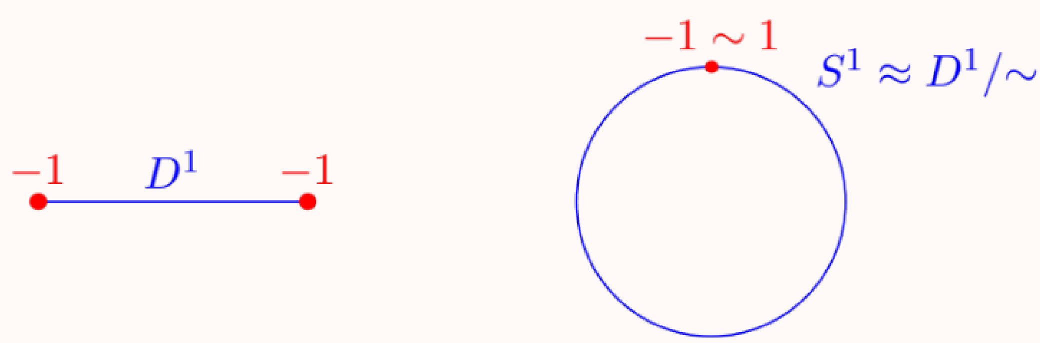
2022.12.28

~~Category Theory (Continued)~~ & Topology

昨天看Cat.看到一半就没在看 Alg. Top 3，今天继续。

Example 57.2.2 (Interval modulo endpoints)

Suppose we take $D^1 = [-1, 1]$ and quotient by the equivalence relation which identifies the endpoints -1 and 1. (Formally, $x \sim y \iff (x = y)$ or $\{x, y\} = \{-1, 1\}$.) In that case, we simply recover S^1:



Observe that a small open neighborhood around $-1 \sim 1$ in the quotient space corresponds to two half-intervals at -1 and 1 in the original space D^1. This should convince you the definition we gave is the right one.

Example 57.2.3 (More quotient spaces)

Convince yourself that:

- Generalizing the previous example, D^n modulo its boundary S^{n-1} is S^n.
- Given a square $ABCD$, suppose we identify segments AB and DC together. Then we get a cylinder. (Think elementary school, when you would tape up pieces of paper together to get cylinders.)
- In the previous example, if we also identify BC and DA together, then we get a torus. (Imagine taking our cylinder and putting the two circles at the end together.)
- Let $X = \mathbb{R}$, and let $x \sim y$ if $y - x \in \mathbb{Z}$. Then $X/\sim$ is S^1 as well.

Definition 57.2.4. Let $A \subseteq X$. Consider the equivalence relation which identifies all the points of A with each other while leaving all remaining points inequivalent. (In other words, $x \sim y$ if $x = y$ or $x, y \in A$.) Then the resulting quotient space is denoted X/A.

So in this notation,

$$D^n/S^{n-1} = S^n.$$

说实话，我感觉这里解释的商集多少有点问题，比如 $\mathbb{Z}/7\mathbb{Z}$ 这种就不能这样看。

A quotient set is what you get when you "divide" a set A by $B \subseteq A$, wherein you set all elements of B to the identity in A. For example, if $A = \mathbb{Z}$ and $B = \{5n \mid n \in \mathbb{Z}\}$, then you're making all multiples of 5 zero for all intents and purposes, so the quotient is $\{0, 1, 2, 3, 4\}$.

这个MSE上的解释又只符合后一种情况了。

我个人认为 X/Y is "X modulo Y" 比较合适，但 modulo 实在不好说，只是符合直觉。比如 $\mathbb{Z}/5\mathbb{Z}$，你可以认为是指对于每个 int modulo 5 取等价类，但 D^n/S^{n-1} 就不好说怎么 modulo 了。当然 X/Y 本来就乱，比如：

(i) $\mathbb{Z}/5\mathbb{Z}$，实指 $\mathbb{Z}/R$，$xRy \Leftrightarrow 5 \mid (x-y)$ (实际上是陪集和商群那块的了)

(ii) D^n/S^{n-1}，实指 D^n/R，$xRy \Leftrightarrow x=y$ or $x, y \in S^{n-1}$

(iii) X/R，$R = \{(x,y) \mid x, y \in X\}$(无序)，实际上 R 就是指一个 equ. relation，只不过写成集合形式。

(iv) X/Y，$Y = X/R$，这个比较离谱，它商出来的是一个关系，就是 R，当然也可以同 (iii) 理解成集合。

当然，(iv) 中 X/R 要理解成 X 的子集的集族，这时也是把 R 视为 X 的一个划分

§57.3 Product topology

Prototypical example for this section: $\mathbb{R} \times \mathbb{R}$ *is* $\mathbb{R}^2$, $S^1 \times S^1$ *is the torus.*

Definition 57.3.1. Given topological spaces X and Y, the **product topology** on $X \times Y$ is the space whose

- Points are pairs (x, y) with $x \in X$, $y \in Y$, and
- Topology is given as follows: the *basis* of the topology for $X \times Y$ is $U \times V$, for $U \subseteq X$ open and $V \subseteq Y$ open.

Remark 57.3.2 — It is not hard to show that, in fact, one need only consider basis elements for U and V. That is to say,

$$\{U \times V \mid U, V \text{ basis elements for } X, Y\}$$

is also a basis for $X \times Y$.

We really do need to fiddle with the basis: in $\mathbb{R} \times \mathbb{R}$, an open unit disk better be open, despite not being of the form $U \times V$.

This does exactly what you think it would.

Example 57.3.5 (More product spaces)

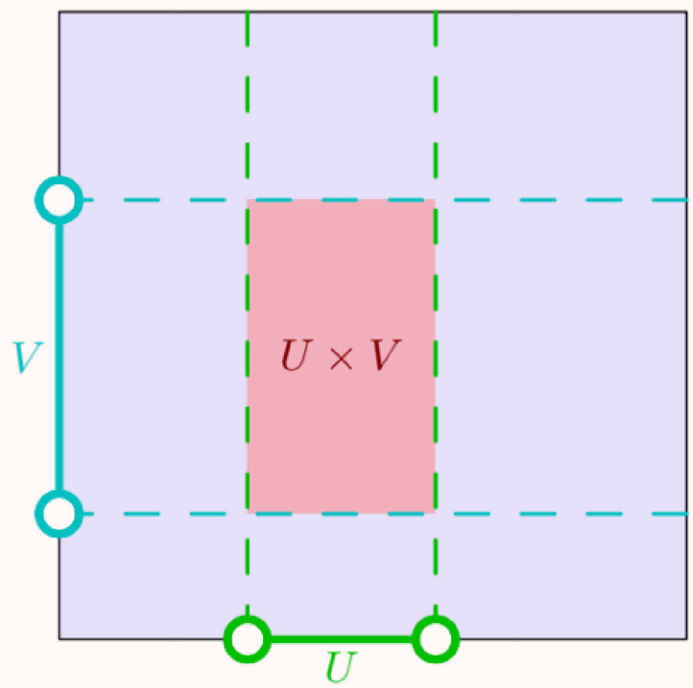
(a) $\mathbb{R} \times \mathbb{R}$ is the Euclidean plane.

(b) $S^1 \times [0, 1]$ is a cylinder.

(c) $S^1 \times S^1$ is a torus! (Why?)

Example 57.3.3 (The unit square)

Let $X = [0,1]$ and consider $X \times X$. We of course expect this to be the unit square. Pictured below is an open set of $X \times X$ in the basis.



关于$S_1 \times S_1 = \mathbb{T}$这个，其实有好几种看法，改天不知道能不能用LaTeX整理一下。