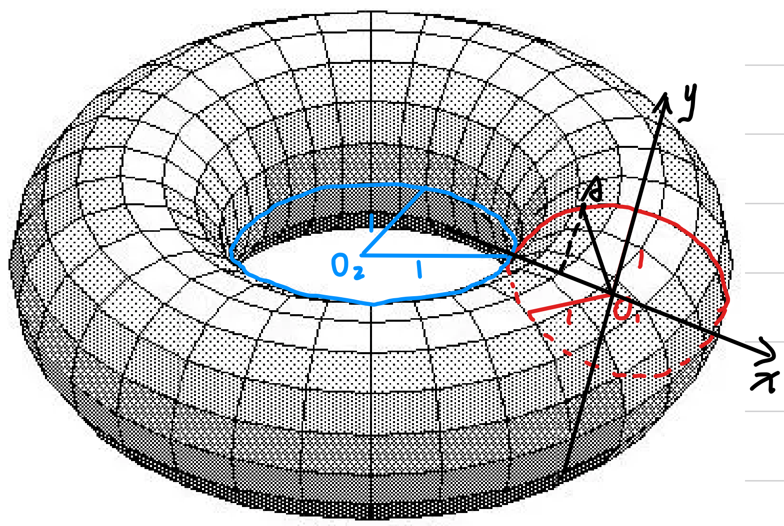
INFORMAL NOTES ON MATHEMATICS
2022.12.29

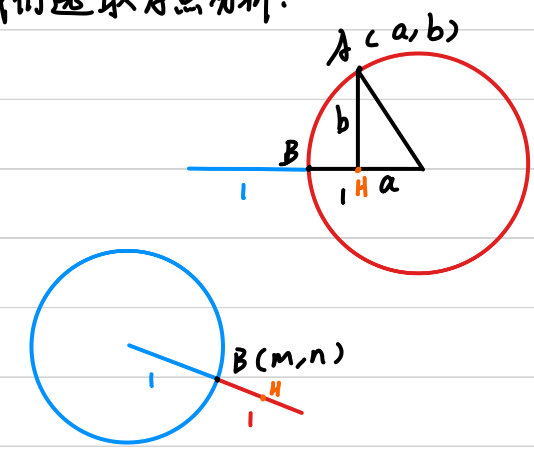
Algebraic Topology & Algebra

下面写一下 $S^1\times S^1=\mathbb{T}$ 的一个理解：

可以把 $\mathbb{T}$ 视为一个竖过来的 S_1 绕一个水平的 S_1 转一周，其边界的轨迹形成的点集。



我们选取A点分析：



$\therefore$ 可构造映射：$f: S^1\times S^1 \to \mathbb{T}$，$(x,y)\in S^1\times S^1$，$x=(m,n)$，$y=(a,b)$

$(x,y) \mapsto (m(2+a),\ n(2+a),\ b)$

显然 f 是一个同胚映射。 $\therefore S^1\times S^1=\mathbb{T}$

§57.4 Disjoint union and wedge sum

Prototypical example for this section: $S^1\vee S^1$ *is the figure eight.*

The disjoint union of two spaces is geometrically exactly what it sounds like: you just imagine the two spaces side by side. For completeness, here is the formal definition.

Definition 57.4.1. Let X and Y be two topological spaces. The **disjoint union**, denoted $X\amalg Y$, is defined by

- The points are the disjoint union $X\amalg Y$, and
- A subset $U\subseteq X\amalg Y$ is open if and only if $U\cap X$ and $U\cap Y$ are open.

> **Exercise 57.4.2.** Show that the disjoint union of two nonempty spaces is disconnected.

Proof: Suppose $T_X=\{\varnothing, U_{X_1}, U_{X_2}, \cdots, X\}$, $T_Y=\{\varnothing, U_{Y_1}, U_{Y_2}, \cdots, Y\}$

It's easy to show all U_{X_i} and U_{Y_i}'s joint (finity) is open in $X\sqcup Y$, so do X, Y

We want to show $X\cup Y\in T_{X\cup Y}$, $X\cup U_{Y_i}\in T_{X\cup Y}$, $Y\cup U_{X_i}\in T_{X\cup Y}$

The above is incorrect. Now rewrite. (Мы нет кноу U_{X_i} с U_{Y_i} ис опен в $X\sqcup Y$)

Шуо шихуа я реалй нет кноу ноу то прув а сет ис опен, Я кноу оно ис обвиоус тхат X и Y

аре опен в $X\sqcup Y$ бат вхэт нуд я до то иоу его. Шит.

我就是шабий，竟然忘了条件，就一句话：

$\because X\sqcup Y$ is open, $\therefore (X\sqcup Y)\cap X=X$, $(X\sqcup Y)\cap Y=Y$ are both open

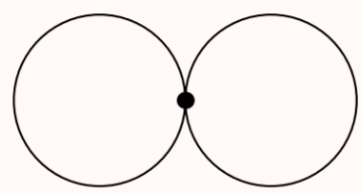
$\therefore X, Y$ are both clopen, which impilies the disconnection. $\square$

More interesting is the wedge sum, where two topological spaces X and Y are fused together only at a single base point.

Definition 57.4.3. Let X and Y be topological spaces, and $x_0 \in X$ and $y_0 \in Y$ be points. We define the equivalence relation $\sim$ by declaring $x_0 \sim y_0$ only. Then the **wedge sum** of two spaces is defined as

$$X \vee Y = (X \amalg Y)/\sim.$$

Example 57.4.4 ($S^1 \vee S^1$ is a figure eight)
Let $X = S^1$ and $Y = S^1$, and let $x_0 \in X$ and $y_0 \in Y$ be any points. Then $X \vee Y$ is a "figure eight": it is two circles fused together at one point.



Abuse of Notation 57.4.5. We often don't mention x_0 and y_0 when they are understood (or irrelevant). For example, from now on we will just write $S^1 \vee S^1$ for a figure eight.

Remark 57.4.6 — Annoyingly, in LaTeX `\wedge` gives $\wedge$ instead of $\vee$ (which is `\vee`). So this really should be called the "vee product", but too late.

确实。之前敲 Boolean algebra时就是这样敲错的。

§57.5 CW complexes

Using this construction, we can start building some spaces. One common way to do so is using a so-called **CW complex**. Intuitively, a CW complex is built as follows:

- Start with a set of points X^0.
- Define X^1 by taking some line segments (copies of D^1) and fusing the endpoints (copies of S^0) onto X^0.
- Define X^2 by taking copies of D^2 (a disk) and welding its boundary (a copy of S^1) onto X^1.
- Repeat inductively up until a finite stage n; we say X is **n-dimensional**.

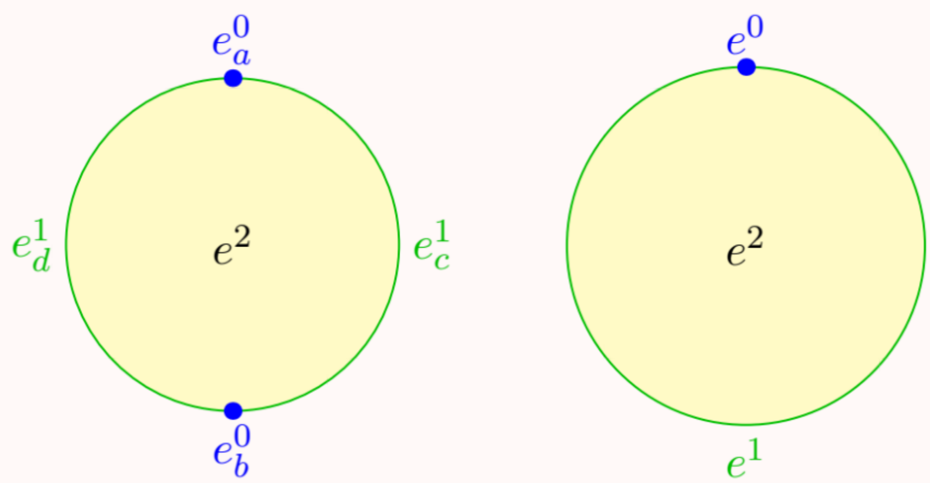
The resulting space X is the CW-complex. The set X^k is called the **k-skeleton** of X. Each D^k is called a **k-cell**; it is customary to denote it by e_α^k where α is some index. We say that X is **finite** if only finitely many cells were used.

Abuse of Notation 57.5.1. Technically, most sources (like [Ha02]) allow one to construct infinite-dimensional CW complexes. We will not encounter any such spaces in the Napkin.

Example 57.5.2 (D^2 with $2+2+1$ and $1+1+1$ cells)

(a) First, we start with X^0 having two points e_a^0 and e_b^0. Then, we join them with two 1-cells D^1 (green), call them e_c^1 and e_d^1. The endpoints of each 1-cell (the copy of S^0) get identified with distinct points of X^0; hence $X^1 \cong S^1$. Finally, we take a single 2-cell e^2 (yellow) and weld it in, with its boundary fitting into the copy of S^1 that we just drew. This gives the figure on the left.

(b) In fact, one can do this using just $1+1+1=3$ cells. Start with X^0 having a single point e^0. Then, use a single 1-cell e^1, fusing its two endpoints into the single point of X^0. Then, one can fit in a copy of S^1 as before, giving D^2 as on the right.



Example 57.5.3 (S^n as a CW complex)

(a) One can obtain S^n (for $n \geq 1$) with just two cells. Namely, take a single point e^0 for X^0, and to obtain S^n take D^n and weld its entire boundary into e^0.

We already saw this example in the beginning with $n=2$, when we saw that the sphere S^2 was the result when we fuse the boundary of a disk D^2 together.

(b) Alternatively, one can do a "hemisphere" construction, by constructing S^n inductively using two cells in each dimension. So S^0 consists of two points, then S^1 is obtained by joining these two points by two segments (1-cells), and S^2 is obtained by gluing two hemispheres (each a 2-cell) with S^1 as its equator.

Definition 57.5.4. Formally, for each k-cell e_α^k we want to add to X^k, we take its boundary S_α^{k-1} and weld it onto X^{k-1} via an **attaching map** $S_\alpha^{k-1} \to X^{k-1}$. Then

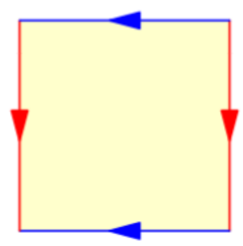
$$X^k = X^{k-1} \amalg \left(\coprod_\alpha e_\alpha^k \right) / \sim$$

where $\sim$ identifies each boundary point of e_α^k with its image in X^{k-1}.

§57.6 The torus, Klein bottle, $\mathbb{RP}^n$, $\mathbb{CP}^n$

§57.6.i The torus

The **torus** can be formed by taking a square and identifying the opposite edges in the same direction: if you walk off the right edge, you re-appear at the corresponding point in on the left edge. (Think *Asteroids* from Atari!)



Thus the torus is $(\mathbb{R}/\mathbb{Z})^2 \cong S^1 \times S^1$.

Note that all four corners get identified together to a single point. One can realize the torus in 3-space by treating the square as a sheet of paper, taping together the left and right (red) edges to form a cylinder, then bending the cylinder and fusing the top and bottom (blue) edges to form the torus.

注意,这个$\mathbb{R}/\mathbb{Z}$是加法运算下构造的陪集,和$\mathbb{Z}/5\mathbb{Z}$类似。我们回头看一眼Hungerford:

Definition 4.1. *Let* H *be a subgroup of a group* G *and* $a, b \in G$. a *is* **right congruent** *to* b **modulo** H, *denoted* $a \equiv_r b \pmod{H}$ *if* $ab^{-1} \in H$. a *is* **left congruent** *to* b **modulo** H, *denoted* $a \equiv_l b \pmod{H}$, *if* $a^{-1}b \in H$.

Theorem 4.2. *Let* H *be a subgroup of a group* G.

(i) *Right [resp. left] congruence modulo* H *is an equivalence relation on* G.

(ii) *The equivalence class of* $a \in G$ *under right [resp. left] congruence modulo* H *is the set* $Ha = \{ha \mid h \in H\}$ *[resp.* $aH = \{ah \mid h \in H\}$*].*

(iii) $|Ha| = |H| = |aH|$ *for all* $a \in G$.

The set Ha is called a **right coset** of H in G and aH is called a **left coset** of H in G. In general it is *not* the case that a right coset is also a left coset (Exercise 2).

Proof: We write $a \equiv b$ for $a \equiv_r b (\mathrm{mod}\, H)$

(i) reflexive: $aa^{-1} = e \in H \Rightarrow a \equiv a$

symmetric: $\because a \equiv b \therefore ab^{-1} \in H \therefore (ab^{-1})^{-1} = ba^{-1} \in H \therefore b \equiv a$

transitive: $\because a \equiv b,\ b \equiv c \therefore ab^{-1} \in H,\ bc^{-1} \in H \therefore ab^{-1}bc^{-1} = ac^{-1} \in H \therefore a \equiv c$

(ii) $\bar{a} = \{x \mid x \equiv a\} = \{x \mid xa^{-1} \in H\} = \{x \mid xa^{-1} = h \in H\} = \{x \mid x = ha \in H\} = \{ha \mid h \in H\}$

(iii) $f: Ha \to H,\ ha \mapsto h$ is a bijection $\square$

Corollary 4.3. *Let H be a subgroup of a group* G.

(i) G *is the union of the right [resp. left] cosets of* H *in* G.

(ii) *Two right [resp. left] cosets of* H *in* G *are either disjoint or equal.*

(iii) *For all* $a, b \in G$, $Ha = Hb \Leftrightarrow ab^{-1} \in H$ *and* $aH = bH \Leftrightarrow a^{-1}b \in H$.

(iv) *If* $\mathfrak{R}$ *is the set of distinct right cosets of* H *in* G *and* $\mathfrak{L}$ *is the set of distinct left cosets of* H *in* G, *then* $|\mathfrak{R}| = |\mathfrak{L}|$.

Proof: (i)~(iii) 是 equ. relation 的性质

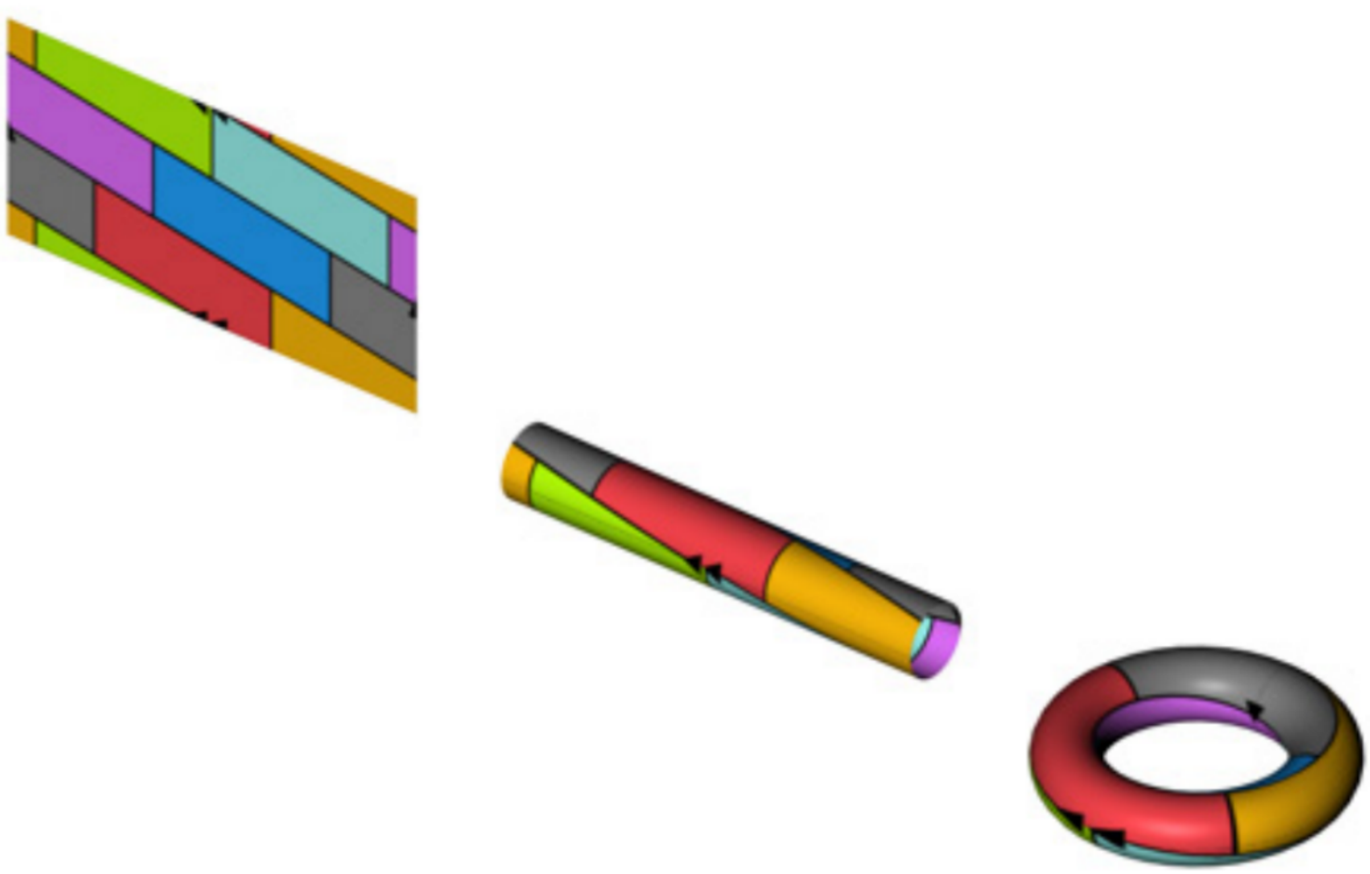
(iv) $\mathcal{R} \to \mathcal{L}$, $Ha \mapsto b^{-1}H$ is a bijection, since $ab^{-1} \to b^{-1}a$ is the same. $\square$

ADDITIVE NOTATION. If H is a subgroup of an additive group, then right congruence modulo H is defined by: $a \equiv_r b \pmod H \Leftrightarrow a - b \in H$. The equivalence class of $a \in G$ is the right coset $H + a = \{h + a \mid h \in H\}$; similarly for left congruence and left cosets.

Definition 4.4. *Let* H *be a subgroup of a group* G. *The* **index of H in G**, *denoted* [G : H], *is the cardinal number of the set of distinct right* [*resp. left*] *cosets of* H *in* G.

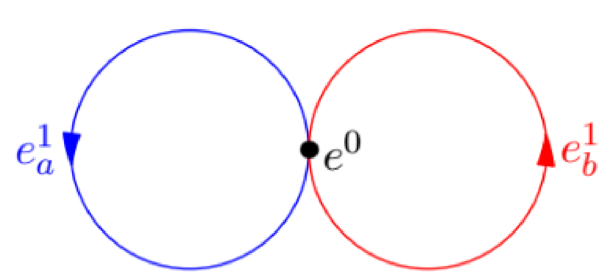
In view of Corollary 4.3 (iv), $[G : H]$ does not depend on whether right or left cosets are used in the definition. Our principal interest is in the case when $[G : H]$ is finite, which can occur even when G and H are infinite groups (for example, $[\mathbf{Z} : \langle m \rangle] = m$ by Introduction, Theorem 6.8(i)). Note that if $H = \langle e \rangle$, then $Ha = \{a\}$ for every $a \in G$ and $[G : H] = |G|$.

好我们暂且回到 Alg. Top. 在此意义下, $\mathbb{R}/\mathbb{Z}$ 很显然可被视为 $[0,1)$, 这样又有了另一个 $S^1 \times S^1 = \mathbb{T}$ 的理解:
$S^1 \times S^1 = [0,1) \times [0,1) = (\mathbb{R}/\mathbb{Z})^2 = \mathbb{T}$. 好, 回到 Napkin:

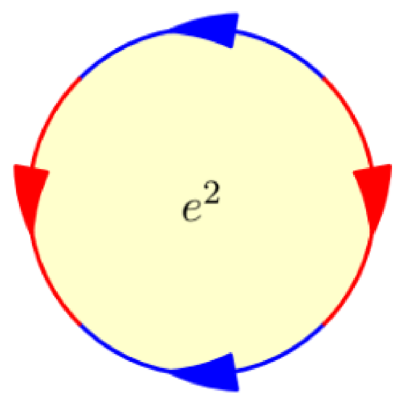


The torus can be realized as a CW complex with

- A 0-skeleton consisting of a single point,
- A 1-skeleton consisting of two 1-cells e^1_a, e^1_b, and



- A 2-skeleton with a single 2-cell e^2, whose circumference is divided into four parts, and welded onto the 1-skeleton "via $aba^{-1}b^{-1}$". This means: wrap a quarter of the circumference around e^1_a, then another quarter around e^1_b, then the third quarter around e^1_a but in the opposite direction, and the fourth quarter around e^1_b again in the opposite direction as before.

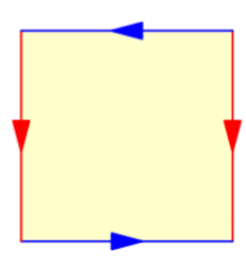


We say that $aba^{-1}b^{-1}$ is the **attaching word**; this shorthand will be convenient later on.

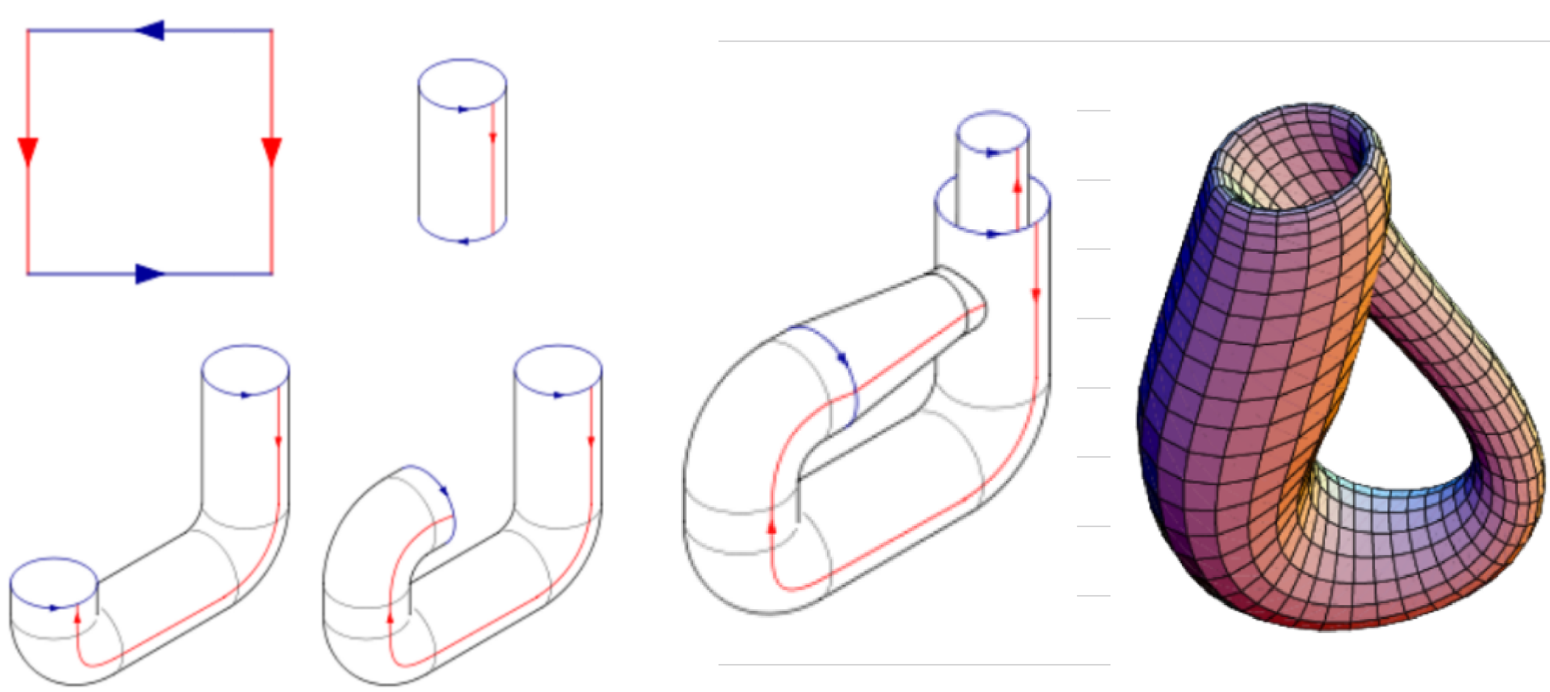
我能理解图，但我还是不明白 attaching word，估计是群论基础缺失。Klein bottle 也是：

§57.6.ii The Klein bottle

The **Klein bottle** is defined similarly to the torus, except one pair of edges is identified in the opposite manner, as shown.



Unlike the torus one cannot realize this in 3-space without self-intersecting. One can tape together the red edges as before to get a cylinder, but to then fuse the resulting blue circles in opposite directions is not possible in 3D. Nevertheless, we often draw a picture in 3-dimensional space in which we tacitly allow the cylinder to intersect itself.



Like the torus, the Klein bottle is realized as a CW complex with

- One 0-cell,
- Two 1-cells e_a^1 and e_b^1, and
- A single 2-cell attached this time via the word $abab^{-1}$.

另外我看这个图竟然看了半小时，我感觉我已经学不成代拓了: