空间中任一物体的万有引力探究

问题起点：球壳对其内质点引力为0。

初等证明：

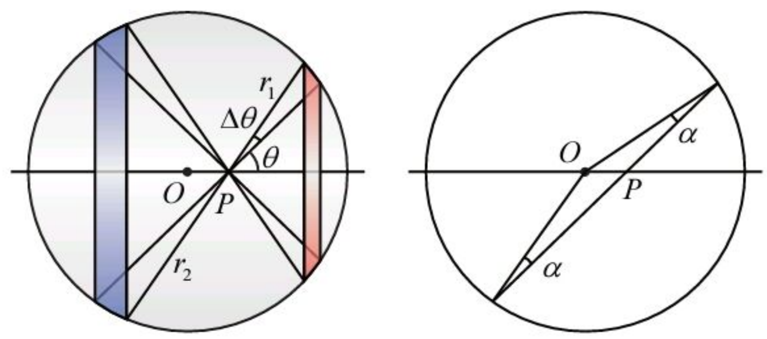


图 2: 几何法

我们现在来证明一个质量面密度为 σ 的均匀球壳在其内部一点 P 产生的引力场为零．如图 2，令球心为 O，并过 OP 作一个轴，这样球面上任意一点都对应一个张角 θ．根据不同的 θ（$0 < \theta < \pi/2$）可将球壳划分为许多对细圆环，每个圆环对应一个 $\Delta\theta$．当 $\Delta\theta \to 0$ 时，如果能证明任意一对细圆环在 P 点产生的引力场都能互相抵消，那么球壳对 P 点的总引力场就为零．

我们先要求出两个圆环的面积，以左图中右边的圆环为例，圆环的周长为 $2\pi r_1 \sin\theta$，当 $\Delta\theta \to 0$ 时，圆环的宽度为 $r_1\Delta\theta/\cos\alpha$（$\alpha$ 的定义见右图），所以圆环的面积等于周长乘以宽度，再乘以面密度 σ 得到右圆环的质量

$$M_1 = 2\pi r_1^2 \sigma \Delta\theta \sin\theta/\cos\alpha \tag{3}$$

同理，左圆环的质量为

$$M_2 = 2\pi r_2^2 \sigma \Delta\theta \sin\theta/\cos\alpha \tag{4}$$

将 M_1, r_1 和 M_2, r_2 分别代入式 2 可得 $g_1 = g_2$ 即两圆环在 P 点产生的引力场大小相等，方向相反，总引力场为零．证毕．

积分法：

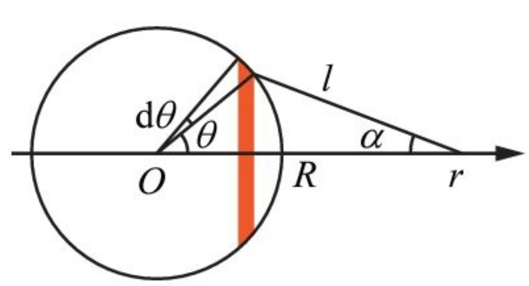


图 3: 积分法

如图 3，令场点离原点 O 的距离为 r（虽然图中 $r > R$，但 $r < R$ 时以下推导同样成立）．和以上推导类似，图中圆环的面积为周长乘以宽度，质量为

$$\mathrm{d}M = 2\pi R\sin\theta \cdot R\,\mathrm{d}\theta \cdot \sigma = 2\pi R^2 \sigma \sin\theta\,\mathrm{d}\theta \tag{5}$$

圆环在场点产生的引力场为

$$\mathrm{d}g = \frac{G\,\mathrm{d}M}{l^2}\cos\alpha \tag{6}$$

其中 $\cos\alpha = (r - R\cos\theta)/l$. 由余弦定理，$l = R^2 + r^2 - 2Rr\cos\theta$. 将 $\mathrm{d}M$，$\cos\alpha$ 和 l 代入式 6 再对 θ 作定积分得

$$g = 2\pi R^2 G\sigma \int_0^{\pi} \frac{(r - R\cos\theta)\sin\theta}{(R^2 + r^2 - 2Rr\cos\theta)^{3/2}}\,\mathrm{d}\theta \tag{7}$$

将上式中的定积分记为 I，使用第一类换元积分法，令 $x = \cos\theta$，得

$$\begin{aligned} I &= \int_{-1}^{1} \frac{r - Rx}{(R^2 + r^2 - 2Rrx)^{3/2}}\,\mathrm{d}x \\ &= \frac{1}{2r}\int_{-1}^{1} \frac{(r^2 - R^2) + (R^2 + r^2 - 2Rrx)}{(R^2 + r^2 - 2Rrx)^{3/2}}\,\mathrm{d}x \\ &= \frac{r^2 - R^2}{2r}\int_{-1}^{1} \frac{1}{(R^2 + r^2 - 2Rrx)^{3/2}}\,\mathrm{d}x + \frac{R^2}{2r}\int_{-1}^{1} \frac{1}{\sqrt{R^2 + r^2 - 2Rrx}}\,\mathrm{d}x \end{aligned} \tag{8}$$

这两个积分可以由“积分表[192]”中的式 2 结合式 1 得到.

$$\begin{aligned} I &= \frac{r^2 - R^2}{2r^2 R}\left(\frac{1}{|r - R|} - \frac{1}{|r + R|}\right) - \frac{1}{2r^2 R}(|r - R| - |r + R|) \\ &= \begin{cases} 2/r^2 & (r > R) \\ 0 & (r < R) \end{cases} \end{aligned} \tag{9}$$

若球分布均匀，设面密度为 σ，则有 $M = 4\pi R^2\sigma$，代入上式得：

$$g = \begin{cases} GM/r^2 & (r > R) \\ 0 & (r < R) \end{cases}$$

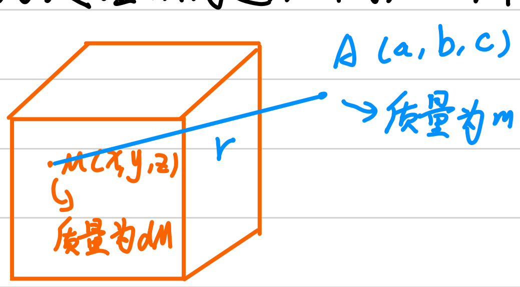
若球分布不均匀，体密度随离球心距离 s 而改变，设变化函数为 $\rho(s)$，则有：

$$g = \begin{cases} GM_0(r)\,r^2, & (r < R) \\ GM/r^2, & (r > R) \end{cases}$$

其中 $M_0(r) = \int_0^r 4\pi s^2 \rho(s)\,ds$

由球壳可知，不是所有物体都能看成质点。实际上除了球体都不能。

为了处理该问题，我们应当采用积分，但不能像球壳那样处理，因此可分成质点：



$$\begin{aligned} G_x &= \int \left|\mathrm{Proj}_x \frac{Gm\,dM}{r^2}\right| \\ &= \int \left|\mathrm{Proj}_x \frac{Gm\rho\,dV}{r^2}\right| \\ &= \iiint Gm\rho \frac{\cos\theta}{r^2}\,dx\,dy\,dz \\ &= Gm\rho \iiint \frac{a-x}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{3/2}}\,dx\,dy\,dz \end{aligned}$$

同理可表示出在 y 方向、在 z 方向上的分量。

比如对于以原点为中心的正方体，其引力大小为：

$$Gm\rho\Bigg[\left(\int_{-N}^{N}\int_{-N}^{N}\int_{-N}^{N} \frac{a-x}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}\,dx\,dy\,dz\right)^2 + \left(\int_{-N}^{N}\int_{-N}^{N}\int_{-N}^{N} \frac{b-y}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}\,dx\,dy\,dz\right)^2$$
$$+ \left(\int_{-N}^{N}\int_{-N}^{N}\int_{-N}^{N} \frac{c-z}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}\,dx\,dy\,dz\right)^2\Bigg]^{\frac{1}{2}}$$

下面作一些验证。我们已知球可视为质点，而球的体积式为：

$$\int_{-1}^{1}\int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}}\int_{-\sqrt{R^2-y^2-z^2}}^{\sqrt{R^2-y^2-z^2}} dx\,dy\,dz$$

将 R=1，(a,b,c) = (2,3,2) 代入验证，我们有：

$$m=\int_{-1}^{1}\int_{-\sqrt{1^2-z^2}}^{\sqrt{1^2-z^2}}\int_{-\sqrt{1^2-z^2-y^2}}^{\sqrt{1^2-z^2-y^2}}\frac{-2+x}{((-2+x)^2+(-3+y)^2+(-2+z)^2)^{3/2}}dxdydz=-0.119521$$

$$n=\int_{-1}^{1}\int_{-\sqrt{1^2-z^2}}^{\sqrt{1^2-z^2}}\int_{-\sqrt{1^2-z^2-y^2}}^{\sqrt{1^2-z^2-y^2}}\frac{-3+y}{((-2+x)^2+(-3+y)^2+(-2+z)^2)^{3/2}}dxdydz=-0.179282$$

$$l=\int_{-1}^{1}\int_{-\sqrt{1^2-z^2}}^{\sqrt{1^2-z^2}}\int_{-\sqrt{1^2-z^2-y^2}}^{\sqrt{1^2-z^2-y^2}}\frac{-2+z}{((-2+x)^2+(-3+y)^2+(-2+z)^2)^{3/2}}dxdydz=-0.119521$$

$$\sqrt{m^2+n^2+l^2}=0.246399=\frac{4\pi}{3}\frac{1^3}{4+9+4}$$

由此知成立。

下面我们来看一篇论文，看看里面有什么别的结论

We first seek the Newtonian gravitational potential of a rectangular solid of uniform density ρ with Newton's universal gravitational constant G. We suppose the rectangular solid has a length L, breadth B and depth D, centered on the origin, then we have the potential

$$\begin{aligned}V(X,Y,Z) &= -G\rho\int_{-D}^{D}\int_{-B}^{B}\int_{-L}^{L}\frac{dx'dy'dz'}{\sqrt{(X-x')^2+(Y-y')^2+(Z-z')^2}} \qquad (1)\\ &= -G\rho\int_{z=-D-Z}^{D-Z}\int_{y=-B-Y}^{B-Y}\int_{x=-L-X}^{L-X}\frac{dxdydz}{\sqrt{x^2+y^2+x^2}}\\ &= -G\rho\int_{x=-L-X}^{L-X}\int_{y=-B-Y}^{B-Y}\left[\ln(z+\sqrt{x^2+y^2+z^2})\right]_{z=-D-Z}^{D-Z}dydx,\end{aligned}$$

where we made the substitution, $x=x'-X$, $y=y'-Y$ and $z=z'-Z$, and completed the integral over the z coordinate. Next, integrating over the y variable, and using $r=\sqrt{x^2+y^2+z^2}$, we find

$$\begin{aligned}V &= -G\rho\int_{x=-L-X}^{L-X}dx\Bigg[\Bigg[y\ln(z+r)+z\ln(y+r) \qquad (2)\\ &\quad -y+x\arctan\frac{y}{x}-x\arctan\frac{yz}{xr}\Bigg]_{z=-D-Z}^{D-Z}\Bigg]_{y=-B-Y}^{B-Y}\end{aligned}$$

and with the integral over x, we achieve our final result

$$\begin{aligned}V(X,Y,Z) &= -G\rho\Bigg[\Bigg[\Bigg[yz\ln(x+r)-\frac{x^2}{2}\arctan\frac{yz}{xr}+xz\ln(y+r)-\frac{y^2}{2}\arctan\frac{xz}{yr} \qquad (3)\\ &\quad +xy\ln(z+r)-\frac{z^2}{2}\arctan\frac{xy}{zr}\Bigg]_{z=-D-Z}^{D-Z}\Bigg]_{y=-B-Y}^{B-Y}\Bigg]_{x=-L-X}^{L-X}\end{aligned}$$

or using $x_1=x, x_2=y, x_3=z$, and $D_1=L, D_2=B, D_3=D$ we can write

$$V(X_1, X_2, X_3) = -G\rho \sum_{j=1}^{3} \left[\sum_{i=1}^{3} \left(\frac{v}{x_i} \ln(x_i + r) - \frac{x_i^2}{2} \arctan \frac{v}{x_i^2 r} \right) \right]_{x_j = -D_j - X_j}^{D_j - X_j} \tag{4}$$

for the gravitational potential of a cuboid mass, where $v = x_1 x_2 x_3$. The pairs of log and arctan terms combine to produce the expected $\frac{1}{r}$ falloff in gravitational potential at large distances. For example, for a $2 \times 2 \times 2m^3$ cube, with $G\rho = 1$, as $x \to \infty$, $yz \ln(x + r) - \frac{x^2}{2} \arctan \frac{yz}{xr} \to -\frac{8}{r}$, as expected for a point source.

即在引力场中做功的潜力

到此为止不过是把积分处理出来了。值得注意的是，它计算了gravitational potential而非引力本身。

B. A lake on the face of a cube

原来低密度可以避免陆地形状的影响

In order to reveal the equipotentials around this object, we can imagine a very low density fluid being added to a face of a $2 \times 2 \times 2$ cube. We specify a very low density fluid in order not to modify the existing field, and by plotting Eq. (3), at a constant potential we find a lake as shown in Fig. 1. A perfect circle is shown for comparison, and we can see how the water is dragged up towards the corners.

到这里比较有趣了。实际上这个形状也的确符合直觉，毕竟正方体引力场的不均主要体现在四角上。

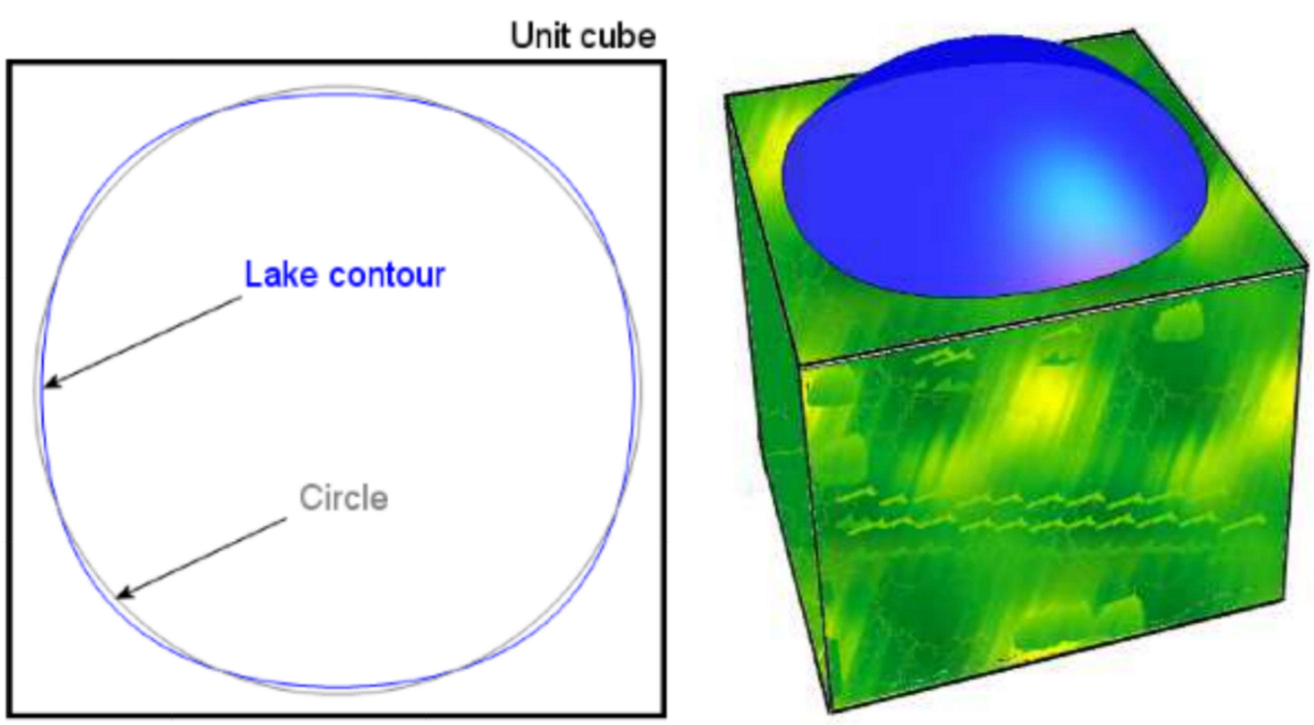


FIG. 1: A lake formed on the surface of a cube. As expected, the edge of the lake is 'pulled up' towards the corners, due to the extra mass present there, but forming a nearly spherical surface.

(实际上，似乎可以就使用这种curvilinear square来计算引力来代替质点。见UTS一份课程讲义。)

If we keep adding the fluid until the edge of the cube is reached, we would have a depth of 0.3346 units over the face, compared to $\sqrt{2} - 1 = 0.4142$ for a spherical surface. If we continue to add fluid until the corners are covered, we would then have a depth of 0.6389 units, above the center of the face, compared to $\sqrt{3} - 1 = 0.7321$ for a spherical surface.

We calculate the gravitational field vectors from

$$\vec{g} = -\nabla V = -\left(\frac{\partial V}{\partial x}e_1 + \frac{\partial V}{\partial y}e_2 + \frac{\partial V}{\partial z}e_3\right). \tag{5}$$

The field vector in the x direction $g_x = -\frac{\partial V}{\partial x}$, can be deduced from Eq. (2), before the last integral is calculated, and hence by the fundamental theorem of calculus we find

$$g_X = G\rho \sum_{j=1}^{3} \left[y \ln(z+r) + z\ln(y+r) - x \arctan\frac{yz}{xr} \right]_{x_j=-D_j-X_j}^{D_j-X_j} \tag{6}$$

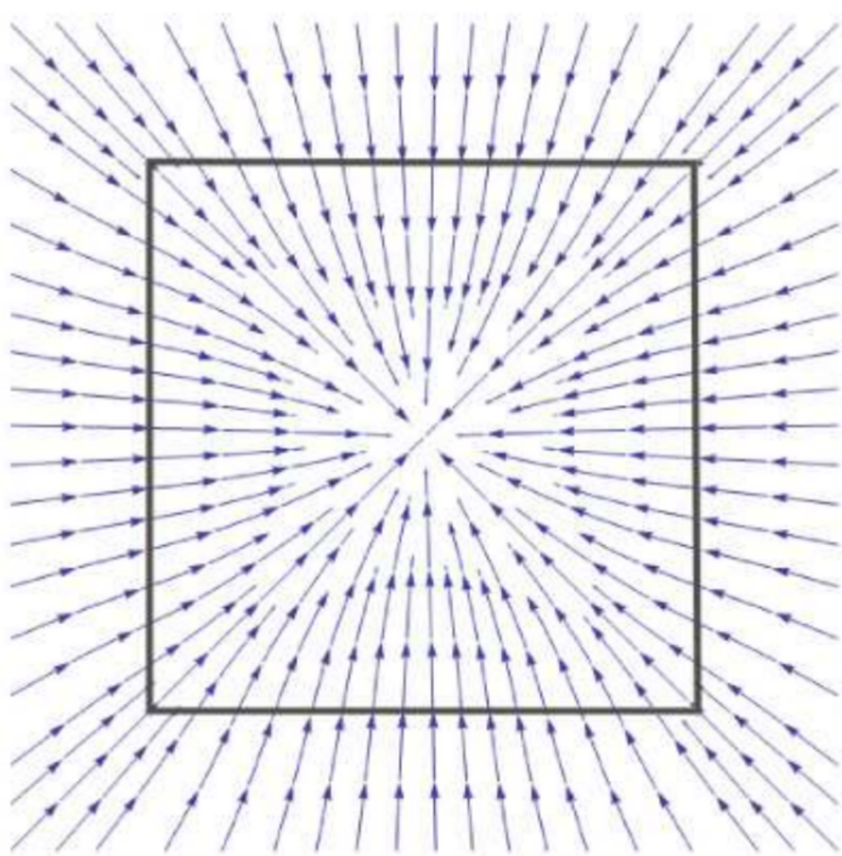
and from symmetry we can also easily deduce the field strengths in the y and z directions, giving the field strength vector $\mathbf{g} = \sum_{i=1}^{3} g_i e_i$, where

$$g_i = G\rho \sum_{b=1}^{3} \left[x_j \ln(x_k+r) + x_k\ln(x_j+r) - x_i \arctan\frac{v}{x_i^2 r} \right]_{x_b=-D_b-X_b}^{D_b-X_b} \tag{7}$$

for distinct i, j, k, where e_i are the unit vectors for the x, y, z coordinate system.

If we look at the changing direction of the field as we move across a face, then we observe that the field vector only points towards the center of the cube at the center of each face, at the corners, and at the center of each edge, which could also be deduced by symmetry arguments, refer Fig. 2.

观察我们的三个分量也能得到该结论。
直观地说，即除3指向中心的分力其余抵销，因此显然要有对称性



D. Orbits around the cube

Could a moon or satellite, orbit this cubic planet? We notice that there is slightly greater gravitational force of attraction over the corners of the cube, and hence an orbiting satellite would significantly couple with the spin of the cube, refer Fig. 3.

If we assume a satellite orbit around the faces and the centre of the edges we can reduce the orbit to the plane, and so need to solve the orbital equations

$$\ddot{x} = g_x\,, \quad \ddot{y} = g_y, \tag{8}$$

refer Eq. (7).

Solving this equation numerically for the specific case, of a satellite orbiting a cube with a side length equal to the diameter of the earth, with an initial height of three earth radii moving in the X direction as shown in Fig. 4, with a velocity of 3.63 km/s.

We find an orbital period of approximately 4.8 hours, and due to the interaction over the corners, the orbit is distorted from a perfect ellipse, creating a fairly rapid precession as shown by the counterclockwise precession of the successive apogees in Fig. 4. The orbit

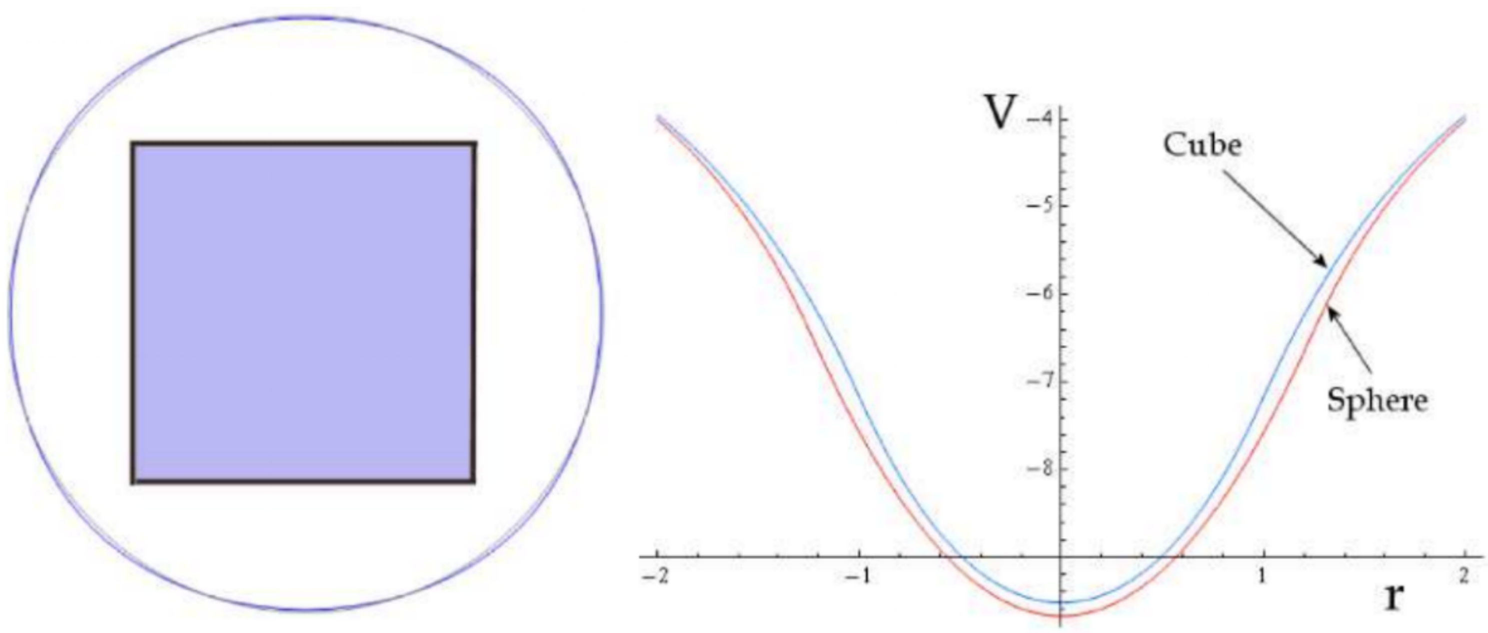


FIG. 3: The equipotential around the equator of the cube. We notice how the field is slightly stronger over the corners, indicated by the equipotential being shifted outwards compared to a perfect circle. When comparing the gravitational potential of a cube to a sphere of the same mass, because the sphere is a more compact object we find a deeper potential well, though the two potentials converge at larger distances as expected.

对于他列出的方程，我先试着拿matlab画着试试。

先介绍一下龙格库塔法（我第一次知道这个的时候还是在大年级）

对ODE $y'(t) = f(y,t)$，令 $h = \Delta t$，$t_n = t_0 + nh$，$y_n = y(t_n)$，则有

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

其中：$k_1 = f(y_n, t_n)$，　$k_2 = f(y_n + h\frac{k_1}{2}, t_n + \frac{h}{2})$

$k_3 = f(y_n + h\frac{k_2}{2}, t_n + \frac{h}{2})$，$k_4 = f(y_n + hk_3, t_n + h)$

那么对于方程，则有：

$$\vec{y}'(t) = f(\vec{y}, t$$

其中

$$\vec{y}'(t) = \begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

首先把龙格库塔法写入一个单独的 matlab 文件，注为 odeRK4.m 方便调用：

```
% 四阶龙格库塔定步长节微分方程
% f(Y, t): 求导函数
% tspan: 二元向量，起始和终止时间
% Y0: 初值（列向量）
% Nt: 时间节点数
function [Y, t] = odeRK4(f, tspan, Y0, Nt)
Nvar = numel(Y0);  % 因变量的个数
dt = (tspan(2) - tspan(1)) / (Nt-1); % 计算步长
Y = zeros(Nvar, Nt); % 预赋值
Y(:, 1) = Y0(:); % 初值
t = linspace(tspan(1), tspan(2), Nt);

for ii=1:Nt-1
    K1 = f(Y(:,ii)            , t(ii)       );
    K2 = f(Y(:,ii)+K1*dt/2    , t(ii)+dt/2 );
    K3 = f(Y(:,ii)+K2*dt/2    , t(ii)+dt/2 );
    K4 = f(Y(:,ii)+K3*dt      , t(ii)+dt    );
    Y(:,ii+1) = Y(:,ii) + dt/6 * (K1+2*K2+2*K3+K4);
end
end
```

列出此条件下方程（不妨设为 $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}$ 且行星在 xOy 平面上运动

$x' = v_x$, $y' = v_y$, $v_x' = g_x$, $v_y' = g_y$ （把二阶拆成一阶）

其中

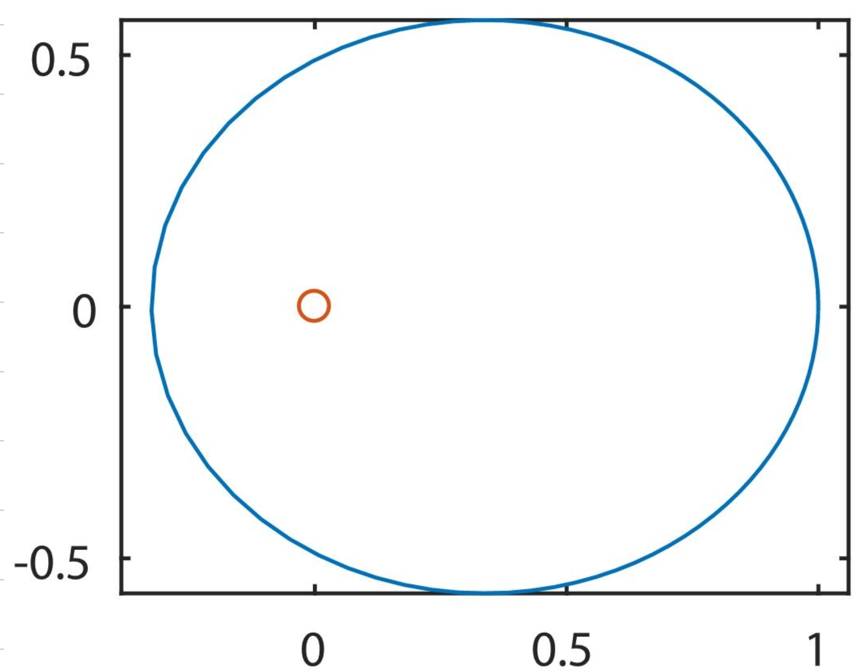
$$g_i = G\rho \sum_{b=1}^{3} \left[x_j \ln(x_k + r) + x_k \ln(x_j + r) - x_i \arctan \frac{v}{x_i^2 r} \right]_{x_b=-D_b-X_b}^{D_b-X_b}$$

```
function keplerRK4
% 参数设定
GM = 1; % 万有引力常数乘以中心天体质量
x0 = 1; y0 = 0; % 初始位置
vx0 = 0; vy0 = 0.7; % 初始速度
tspan = [0; 4]; % 总时间和步数
Nt = 100; % 步数

Y0 = [x0; y0; vx0; vy0]; % 因变量初值
f = @(Y, t)fun(Y, t, GM);
[Y,~] = odeRK4(f, tspan, Y0, Nt);

% 画图
figure; hold on;
plot(Y(1,:), Y(2,:));
scatter(0, 0);
axis equal;
end

function Y1 = fun(Y, ~, GM)
```

```
% 因变量
x = Y(1); y = Y(2);
vx = Y(3); vy = Y(4);
Y1 = zeros(4,1); % 预赋值
Y1(1) = vx;
Y1(2) = vy;
temp = -GM /(x^2 + y^2)^(3/2);
Y1(3) = temp * x;
Y1(4) = temp * y;
end
```

列错了，这是球体的。懒得改了。

唔，gx、gy的式子被写成球形的了，但思路是一样的，只要改最后那几行就好了。
这个代码直接画出椭圆，实际验证了开普勒定律：



就这样吧，以下是论文的结论及图：

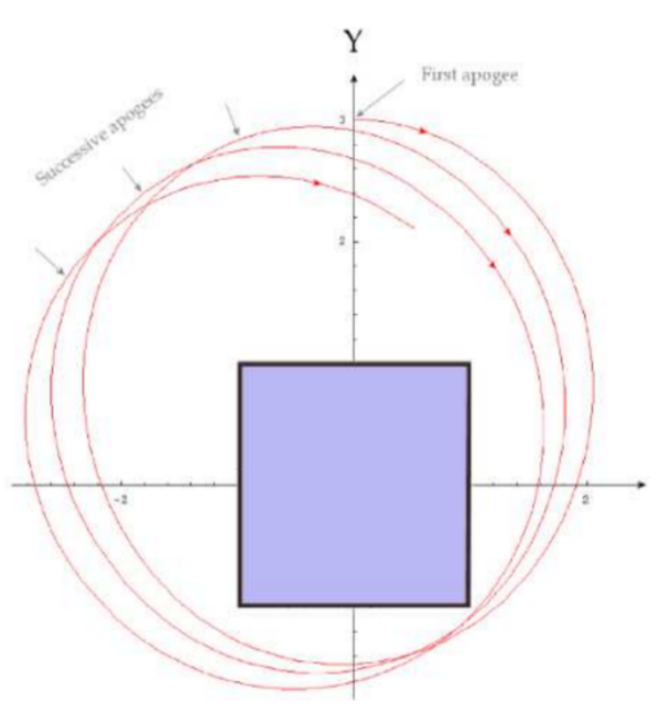


FIG. 4: A satellite orbiting around the equator of a cube with a side length equal to the diameter of the earth. Beginning with an satellite orbital radius of three earth radii and a velocity of 3.63 km/s, we find a period of approximately 4.8 hours, but with an orbit that precesses fairly rapidly, as shown by the counterclockwise movement of successive apogees.

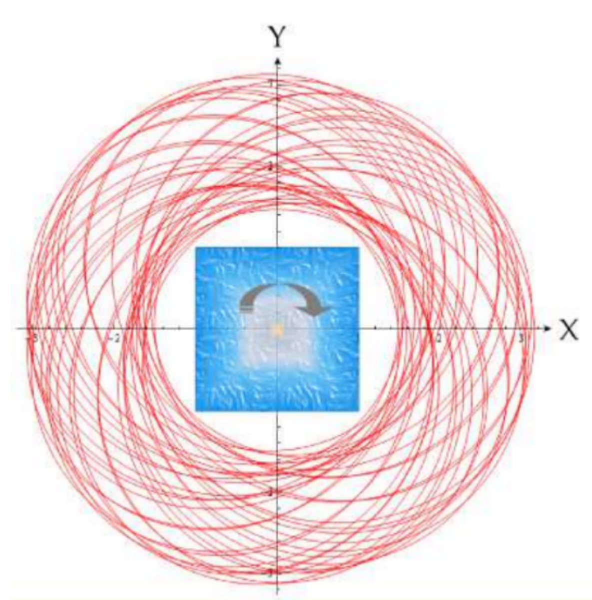


FIG. 5: Following from the previous figure except that the central cube is now rotating with a 24 hour day, in conjunction with the 4.8 hour satellite orbit. Both rotations are clockwise, but the irregular perturbation of the satellite by the cube creating a continually varying trajectory.

补充：把我们上面的直角坐标系的结论写成球坐标：

$$G_x = Gm\rho \iiint \frac{a-x}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}\,dx\,dy\,dz$$

$$= Gm\rho \iiint \frac{(r_0 \sin\varphi_0 \cos\theta_0 - r\sin\varphi\cos\theta)\cdot r^2 \sin\varphi \, dr\, d\varphi\, d\theta}{[(r_0\sin\varphi_0\cos\theta_0 - r\sin\varphi\cos\theta)^2 + (r_0\sin\varphi_0\sin\theta_0 - r\sin\varphi\sin\theta)^2 + (r_0\cos\varphi_0 - r\cos\varphi)^2]^{\frac{3}{2}}}$$

代入仍然成立。但实际上这样处理太繁琐了，估计采用环面逼近也行（?）。

似乎球坐标在处理非球类还是比较不妥。