Category Theory (Continued)

Example 60.2.8 (Posets are categories)

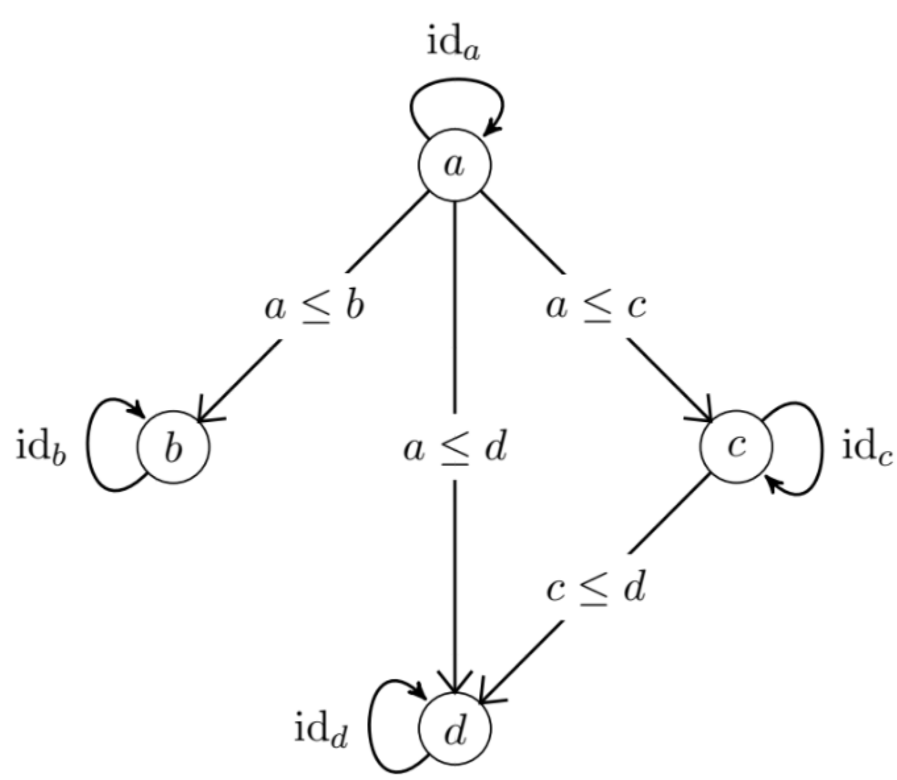
Let $\mathcal{P}$ be a partially ordered set. We can construct a category P for it as follows:

- The objects of P are going to be the elements of $\mathcal{P}$.
- The arrows of P are defined as follows:
  - For every object $p \in P$, we add an identity arrow id_p, and
  - For any pair of distinct objects $p \leq q$, we add a single arrow $p \to q$.

  There are no other arrows.
- There's only one way to do the composition. What is it?

这其实是一个并不陌生的例子，之前在Geek学院时听的Cat.就有。≤也可作为arrow

For example, for the poset $\mathcal{P}$ on four objects $\{a, b, c, d\}$ with $a \leq b$ and $a \leq c \leq d$, we get:



This illustrates the point that

The arrows of a category can be totally different from functions.

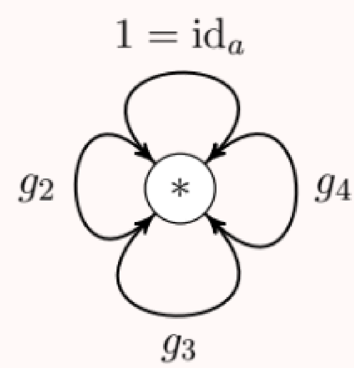
Definition 60.2.5. An arrow $A_1 \xrightarrow{f} A_2$ is an **isomorphism** if there exists $A_2 \xrightarrow{g} A_1$ such that $f \circ g = \mathrm{id}_{A_2}$ and $g \circ f = \mathrm{id}_{A_1}$. In that case we say A_1 and A_2 are **isomorphic**, hence $A_1 \cong A_2$.

Question 60.2.9. Check that no two distinct objects of a poset are isomorphic.

Obviously.

Example 60.2.10 (Important: groups are one-Object categories)

A group G can be interpreted as a category $\mathcal{G}$ with one object $*$, all of whose arrows are isomorphisms.



As [Le14] says:

> The first time you meet the idea that a group is a kind of category, it's tempting to dismiss it as a coincidence or a trick. It's not: there's real content. To see this, suppose your education had been shuffled and you took a course on category theory before ever learning what a group was. Someone comes to you and says:
>
> "There are these structures called 'groups', and the idea is this: a group is what you get when you collect together all the symmetries of a given thing."
>
> "What do you mean by a 'symmetry'?" you ask.
>
> "Well, a symmetry of an object X is a way of transforming X or mapping X into itself, in an invertible way."
>
> "Oh," you reply, "that's a special case of an idea I've met before. A category is the structure formed by *lots* of objects and mappings between them – not necessarily invertible. A group's just the very special case where you've only got one object, and all the maps happen to be invertible."

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> Yes, you can. Define a category M with just one formal object say $ob(M) = \{X\}$. Let G be a group. Define $Mor(X, X)$ = underlying set of G, and composition of morphisms in $Mor(X, X)$ by the binary operation on G. The identity morphism on X is just the identity element in G. Then you can verify that all axioms of a category are satisfied by M. Since each element in G has an inverse, note, moreover, that every element in $Mor(X, X)$ is an isomorphism. In

这其实有点抽象，不过也恰恰说明了arrow可以是很怪的东西。你定义的Mor并不一定符合直觉上Mor的含义，也不一定对obj执行了什么操作或是某种relation.

下面跳回 product.

Of course, we can define products of more than just one object. Consider a set of objects $(X_i)_{i\in I}$ in a category $\mathcal{A}$. We define a **cone** on the X_i to be an object A with some "projection" maps to each X_i. Then the **product** is a cone P which is "universal" in the same sense as before: given any other cone A there is a unique map $A \to P$ making the diagram commute. In short, a product is a "universal cone".

The picture of this is

cone 似乎真翻译成锥

$$\begin{tikzcd}
 & & A \arrow[d, "\exists! f"] \arrow[lld, bend right] \arrow[ld, bend right] \arrow[rd, bend left] \arrow[rrd, bend left] & & \\
 & & P \arrow[lld] \arrow[ld] \arrow[rd] \arrow[rrd] & & \\
X_1 & X_2 & & X_3 & X_4
\end{tikzcd}$$

See also Problem 60C.

One can also do the dual construction to get a **coproduct**: given X and Y, it's the object $X+Y$ together with maps $X \xrightarrow{\iota_X} X+Y$ and $Y \xrightarrow{\iota_Y} X+Y$ (that's Greek iota, think inclusion) such that for any object A and maps $X \xrightarrow{g} A$, $Y \xrightarrow{h} A$ there is a unique f for which

$$\begin{tikzcd}
X \arrow[rd, "\iota_X"'] \arrow[rrd, bend left, "g"] & & \\
 & X+Y \arrow[r, "\exists! f"] & A \\
Y \arrow[ru, "\iota_Y"] \arrow[rru, bend right, "h"'] & &
\end{tikzcd}$$

Exercise 60.4.7. Describe the coproduct in Set.

Predictable terminology: a coproduct is a universal **cocone**.

Spoiler alert later on: this construction can be generalized vastly to so-called "limits", and we'll do so later on.

集合的余积是无交并。但这个无交并不是我曾经理解的那种（На я щи дуии ициш доу иищуье цуш На ишен ши щас дань.）

这里无交并指给A.B中每个元素配上一个"标识元素"使其原本的重复元素被视为不同的。由此可知，一定有$X\to X+Y$的一个map ι_X，而f的存在性也是显然的。g确定，因而f唯一。直观来说，如果记$X\to X+Y$, $x\mapsto(x,*)$，那么$f:(x,*)\mapsto g(x)$，即忽略"标识"。Y同理。下面放上无交并的定义防止忘了

The disjoint union of two sets A and B is a binary operator that combines all distinct elements of a pair of given sets, while retaining the original set membership as a distinguishing characteristic of the union set. The disjoint union is denoted

$$A \bigcup^{*} B = (A \times \{0\}) \bigcup (B \times \{1\}) \equiv A^{*} \bigcup B^{*}, \tag{1}$$

where $A \times S$ is a Cartesian product. For example, the disjoint union of sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4\}$ can be computed by finding

$$A^{*} = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0)\} \tag{2}$$
$$B^{*} = \{(1, 1), (2, 1), (3, 1), (4, 1)\}, \tag{3}$$

so

$$A \bigcup^{*} B = A^{*} \bigcup B^{*} \tag{4}$$
$$= \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (1, 1), (2, 1), (3, 1), (4, 1)\}. \tag{5}$$

The notion of "injective" doesn't make sense in an arbitrary category since arrows need not be functions. The correct categorical notion is:

Definition 60.5.1. A map $X \xrightarrow{f} Y$ is **monic** (or a monomorphism) if for any commutative diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

we must have $g = h$. In other words, $f \circ g = f \circ h \implies g = h$.

Definition 60.5.6. A map $X \xrightarrow{f} Y$ is **epic** (or an epimorphism) if for any commutative diagram

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} A$$

we must have $g = h$. In other words, $g \circ f = h \circ f \implies g = h$.

This is kind of like surjectivity, although it's a little farther than last time. Note that in concrete categories, surjective $\implies$ epic.