Category Theory (Continued)

Example 61.2.2 (Free and forgetful functors)

Note that these are both informal terms, and don't have a rigid definition.

(a) We talked about a **forgetful functor** earlier, which takes the underlying set of a category like Vect_k. Let's call it $U : \mathsf{Vect}_k \to \mathsf{Set}$.

Now, given a map $T : V_1 \to V_2$ in Vect_k, there is an obvious $U(T) : U(V_1) \to U(V_2)$ which is just the set-theoretic map corresponding to T.

Similarly there are forgetful functors from Grp, CRing, etc., to Set. There is even a forgetful functor $\mathsf{CRing} \to \mathsf{Grp}$: send a ring R to the abelian group $(R, +)$. The common theme is that we are "forgetting" structure from the original category.

(b) We also talked about a **free functor** in the example. A free functor $F : \mathsf{Set} \to \mathsf{Vect}_k$ can be taken by considering $F(S)$ to be the vector space with basis S. Now, given a map $f : S \to T$, what is the obvious map $F(S) \to F(T)$? Simple: take each basis element $s \in S$ to the basis element $f(s) \in T$.

Similarly, we can define $F : \mathsf{Set} \to \mathsf{Grp}$ by taking the free group generated by a set S.

Remark 61.2.3 — There is also a notion of "injective" and "surjective" for functors (on arrows) as follows. A functor $F \colon \mathcal{A} \to \mathcal{B}$ is **faithful** (resp. **full**) if for any A_1, A_2, $F \colon \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{B}}(FA_1, FA_2)$ is injective (resp. surjective).[a]

We can use this to give an exact definition of concrete category: it's a category with a faithful (forgetful) functor $U \colon \mathcal{A} \to \mathsf{Set}$.

[a] Again, experts might object that $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ or $\mathrm{Hom}_{\mathcal{B}}(FA_1, FA_2)$ may be proper classes instead of sets, but I am assuming everything is locally small.

之前很多次提到了 concrete category 都没去理它。直觉上就是比较符合认识的对象组成的 Cat. 像 Grp. Cring, 首先 obj 都是基于集合的对象，然后 Mor 就是一般脑子里想到的 mapping。

标准定义就是说，对于 concrete cat. 中两个 obj 之间的 arrow，都唯一对应着 Set 中这两个 obj 的 underlying sets 之间的态射；这是显然的（从例子来看），很显然无论如何元素对应关系 preserve. 至于 forgetful, 这更显然。

Example 61.2.4 (Functors from $\mathcal{G}$)

Let G be a group and $\mathcal{G} = \{*\}$ be the associated one-object category.

(a) Consider a functor $F : \mathcal{G} \to \mathsf{Set}$, and let $S = F(*)$. Then the data of F corresponds to putting a *group action* of G on S.

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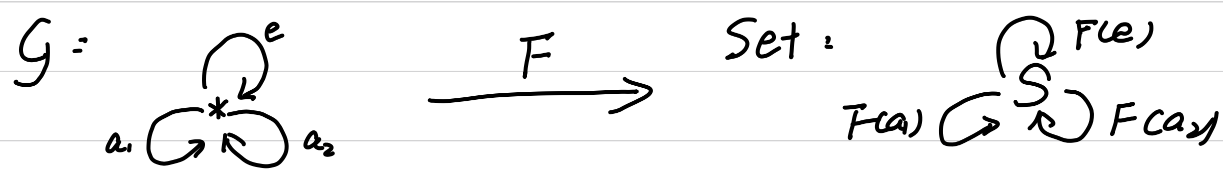

不妨看作右群作用。 $\mathrm{Hom}_{\mathcal{G}}(*,*)$ = G's underlying set $= \{e, a_1, a_2 \cdots\}$

$\mathrm{Hom}_{Set}(S, S) = \{F(e), F(a_1), \cdots\}$ → 映射的集合

那么对于某个右群作用 $\alpha: S\times G \to S$，$(x, a) \mapsto x\cdot a$ $(x\in S, a\in G)$，可以视为：

$(x,a) \mapsto (F(a))(x)$，又易知 $F(e) = id_S$，那么 $(x, e) \mapsto (F(e))(x)$，i.e. $id_S(x) = x$，

符合群作用条件，(resp. 左群作用)，因此 F 可看作一个群作用。

> (c) Let H be a group and construct $\mathcal{H}$ the same way. Then functors $\mathcal{G} \to \mathcal{H}$ correspond to homomorphisms $G \to H$.

接上一问想还是很直觉的。唯一要想的就是保证 $\sigma(a_1 \times_G a_2) = \sigma(a_1) \times_H \sigma(a_2)$

Цаолэ，我突然想起来 functor 的定义：

- The functor respects composition: if $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ are arrows in $\mathcal{A}$, then $F(g\circ f) = F(g)\circ F(f)$.

那这太显然了，毕竟群视为 Cat.，元素的 × 就是复合，functor 导出的 σ 肯定是同态。

明天要起早，就此打住。(离 natural trans. 还有距离，唉)