Informal Notes on Mathematics

2023.03.23

组合数学

1. 组合恒等式

(1) $\sum_{i=0}^{n} C_n^i = 2^n$ （二项式定理）

(2) $\sum_{i=r}^{n} C_i^r = C_{n+1}^{r+1}$ $C_{n+1}^{r+1} \Rightarrow \{1,2,\cdots,n+1\}$ 中选 $r+1$ 个元素

按最大元素分类，若最大为 $i+1$，则有 C_i^r 种选法，再对 i 求和即 LHS

(3) $\sum_{i=0}^{r} C_m^i C_n^{r-i} = C_{n+m}^r$ n 个红球，m 个蓝球，从中选 r 个球 $\Rightarrow$ RHS

LHS：按红球、蓝球个数分类 i 个红 + $r-i$ 个蓝，对 i 求和 $\Rightarrow$ LHS

(4) $C_n^r C_{n-r}^{k-r} = C_n^k C_k^r$

题 23. 若非空集合 $A \subset \{1,2,3,\ldots,n\}$ 满足 $|A| \le \min_{x\in A} x$. 则称 A 为 n 级好集合. 记 a_n 为 n 级好集合的个数. 求证: 对一切正整数 n, 都有 $a_{n+2} = a_{n+1} + a_n + 1$.

解：$|A| \le \min\limits_{x\in A} x$

方法 1：对于 $|A| = k$，$\min\limits_{x\in A} x \ge k$，$A$ 的元素取值于 $\{k, k+1, \cdots, n\}$

若 $n-k+1 < k$，则有 C_{n-k+1}^k 个 k 元好集合，若 $k \le n-k+1 \Rightarrow k \le \frac{n+1}{2}$，$a_n = \sum_{k=1}^{[\frac{n+1}{2}]} C_{n-k+1}^k$

当 n 偶：设 $n=2m$，$a_{2m+2} = \sum_{k=1}^{m+1} C_{2m+2-k+1}^k = C_{2m+2}^1 + C_{2m+2}^2 + \cdots + C_{m+2}^{m+1}$

用 $C_n^i = C_{n-1}^i + C_{n-1}^{i-1}$ 递推

$= (C_{2m+1}^1 + C_{2m+1}^0) + (C_{2m}^2 + C_{2m}^1) + \cdots + (C_{m+1}^{m+1} + C_{m+1}^m)$

$= \underbrace{(C_{2m+1}^1 + C_{2m}^2 + \cdots + C_{m+1}^{m+1})}_{a_{2m+1}} + 1 + \underbrace{C_{2m}^1 + C_{2m-1}^2 + \cdots + C_{m+1}^m}_{a_{2m}}$

$= a_{2m+1} + a_{2m} + 1$, ✓

当 n 奇：设 $n = 2m-1$，则 $a_{2m+1} = C_{2m+1}^1 + C_{2m}^2 + \cdots + C_{m+1}^{m+1}$

$= (C_{2m}^1 + C_{2m}^0) + (C_{2m-1}^2 + C_{2m-1}^1) + \cdots + (C_m^{m+1} + C_m^m)$

$= (C_{2m}^1 + C_{2m-1}^2 + \cdots + C_{m+1}^m) + 1 + (C_{2m-1}^1 + C_{2m-2}^2 + \cdots + C_m^m)$

$= a_{2m} + a_{2m-1} + 1$, ✓

方法 2：$a_{n+2} = a_{n+1} + a_n + 1$

1° 若 $n+2 \notin A$，这样的好集合个数为 a_{n+1}

2° 若 $n+2 \in A$，$A \setminus \{n+2\} = A'$，$|A| \le \min\limits_{x\in A} x$

(1) $|A| \ge 2$，$|A'| = |A| - 1$ 且 $\min\limits_{x\in A} x$ 不变，$|A'| \le \min\limits_{x\in A'} x - 1$

作映射：$A' \to \underbrace{A-1}_{\to A''} = \{x-1 \mid x \in A'\}$，$|A''| \le \min\limits_{x\in A''} x$

(2) $|A| = 1 \Rightarrow A = \{n+2\}$ ✓

$\therefore a_{n+2} = a_{n+1}\ (n+2 \notin A) + a_n\ (n+2 \in A, |A| \ge 2) + 1\ (A = \{n+2\})$

题 24. 一个无现场的整数列 $a_0, a_1, \ldots, a_n$ (不需要两两不同) 满足如下性质: 对任意整数 $i \geq 0$ 都有 $0 \leq a_i \leq i$, 并且对任意整数 $k \geq 0$ 都有

$$C_k^{a_0} + C_k^{a_1} + \ldots + C_k^{a_k} = 2^k$$

证明: 任意正整数 $N \geq 0$ 都会出现在这个数列里 (即对任意 $N \geq 0$, 都存在一个 $i \geq 0$ 满足 $a_i = N$).

解: 1° $a_k = k \Rightarrow$ 二项式定理　$k=0$: $C_0^{a_0} = 2^0 = 1 \Rightarrow a_0 = 0$

$k=1$: $C_1^0 + C_1^{a_1} = 2 \Rightarrow a_1 = 0$ 或 1　$k=2$: $C_2^0 + C_2^1 + C_2^{a_2} = 4 \Rightarrow a_2 = 0$ 或 2

$k=3$: $a_3 = 1$ 或 2, $k=4$: $a_4 = 1$ 或 3　根据此规则用归纳法

证明: 归纳证明 $a_1, \cdots, a_k$ 有如下结构:

$0, 1, \cdots, r-1$ ← r 项 ①

$0, 1, \cdots, k-r$ ← $k-r+1$ 项 ②

$r \geq k-r+1 \Leftrightarrow 2r \geq k+1$

$k=0$, ✓, 设 $k=m$ 成立, 对于 $k=m+1$

$$C_{m+1}^{a_0} + C_{m+1}^{a_1} + \cdots + C_{m+1}^{a_m} + C_{m+1}^{a_{m+1}} = 2^{m+1}$$

$$\text{LHS} = (C_{m+1}^0 + C_{m+1}^1 + \cdots + C_{m+1}^{r-1}) + (C_{m+1}^0 + C_{m+1}^1 + \cdots + C_{m+1}^{m-r}) + C_{m+1}^{a_{m+1}}$$

$$= (C_{m+1}^0 + C_{m+1}^1 + \cdots + C_{m+1}^{r-1} + C_{m+1}^{r+1} + \cdots + C_{m+1}^{m} + C_{m+1}^{m+1}) + C_{m+1}^{a_{m+1}}$$

加上 C_{m+1}^r 之后括号部分即为二项式定理

$\therefore C_{m+1}^{a_{m+1}} = C_{m+1}^r \Rightarrow a_{m+1} = r$ (在①中) 或 $m+1-r$ (在②中) □

母函数:

定义: $\{a_n\}_{n=0}^{\infty}$ 定义母函数 $f(x) = \sum_{n=0}^{\infty} a_n x^n$

例26: 求 $x_1 + x_2 + \cdots + x_k = n$ 非负整数解个数

正常做法: 隔板法 $(n+k)$ 个球中间放 $k-1$ 个隔板 $\Rightarrow C_{n+k-1}^{k-1}$

母函数做法: 考虑 $f(x) = (1+x+x^2+\cdots)^k = (\sum_{i=0}^{\infty} x^i)^k$　$f(x)$ 的 x^n 项系数即原方程非负整数解个数

(理由: x^n 项实为从 k 个 $(1+x+x^2+\cdots)$ 中分别选出一项, 这些项幂的和为 n)

当 $|x| < 1$, $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, $f(x) = (\frac{1}{1-x})^k = \frac{1}{(1-x)^k}$

对 $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ 两边求 k 次导

$$\frac{(k-1)!}{(1-x)^k} = (k-1)! + \frac{(k-1+1)!}{1!}x + \frac{(k-1+2)!}{2!}x^2 + \cdots + \frac{(k-1+n)!}{n!}x^n + \cdots$$

$$\Rightarrow a_n = \frac{(k+n-1)!}{n!\,(k-1)!} = C_{n+k-1}^n$$

例27: 对 $1 \times n$ 用 R、Y、B 三种颜色染色, R格子有偶数个, B格子只有一个, 求方案数

$$\text{考虑 } f(x) = \underbrace{(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)}_{\text{R格子}} \cdot \underbrace{(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)}_{\text{B格子}} \cdot \underbrace{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)}_{\text{Y格子}}$$

(除以 $k!$ 是除以 k 个同色格的顺序)

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$　$\therefore f(x) = \frac{e^x + e^{-x}}{2}(e^x - 1)e^x = \frac{1}{2}(e^{3x} - e^{2x} + e^x - 1)$

其中 x^n 项系数为 $\frac{1}{2}\left(\frac{3^n}{n!}-\frac{2^n}{n!}+\frac{1}{n!}\right)$

$a_n=\frac{1}{2}\left(\frac{3^n}{n!}-\frac{2^n}{n!}+\frac{1}{n!}\right)A_n^n$ → n个格子顺序) $=\frac{1}{2}(3^n-2^n+1)$

例28：有n枚硬币，甩桌上，若正面朝上则放进存钱罐，其余重复此操作，求操作次数的期望

设期望为 e_n，有i个反面向上概率 $C_n^i 2^{-n}$ ⇒ $\boxed{e_n=\sum_{i=0}^{n}\frac{C_n^i}{2^n}(e_i+1) \checkmark =1+2^{-n}\sum_{i=0}^{n}C_n^i e_i}$

令 $f(x)=\sum_{n=1}^{\infty}\frac{e_n}{n!}x^n$，将 e_n 递推代入

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n!}\left(1+2^{-n}\sum_{i=0}^{n}C_n^i e_i\right)$$

$$=e^x-1+\sum_{n=1}^{\infty}\sum_{i=1}^{n}C_n^i e_i\frac{(\frac{x}{2})^n}{n!}$$

$$=e^x-1+\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}C_n^i e_i\frac{(\frac{x}{2})^n}{n!}$$ （交换求和符号）

$$=e^x-1+\sum_{i=1}^{\infty}e_i\frac{1}{i!}\sum_{n=i}^{\infty}\frac{(x/2)^n}{(n-i)!}$$

$$=e^x-1+\sum_{i=1}^{\infty}\frac{e_i(\frac{x}{2})^i}{i!}\underbrace{\sum_{n=i}^{\infty}\frac{(x/2)^{n-i}}{(n-i)!}}_{\to e^{\frac{x}{2}}}$$

$$=e^x-1+\left(\sum_{i=1}^{\infty}\frac{e_i}{i!}\left(\frac{x}{2}\right)^i\right)e^{\frac{x}{2}}=e^x-1+e^{\frac{x}{2}}f\left(\frac{x}{2}\right)$$

$e^{-x}f(x)=1-e^{-x}+e^{-\frac{x}{2}}f\left(\frac{x}{2}\right)$，令 $g(x)=e^{-x}f(x)$

$g(x)=1-e^{-x}+g\left(\frac{x}{2}\right)$　$g(0)=e^0f(0)=0$

$g(x)=1-e^{-x}+g\left(\frac{x}{2}\right)=(1-e^{-x})+(1-e^{-\frac{x}{2}})+g\left(\frac{x}{4}\right)=\cdots$

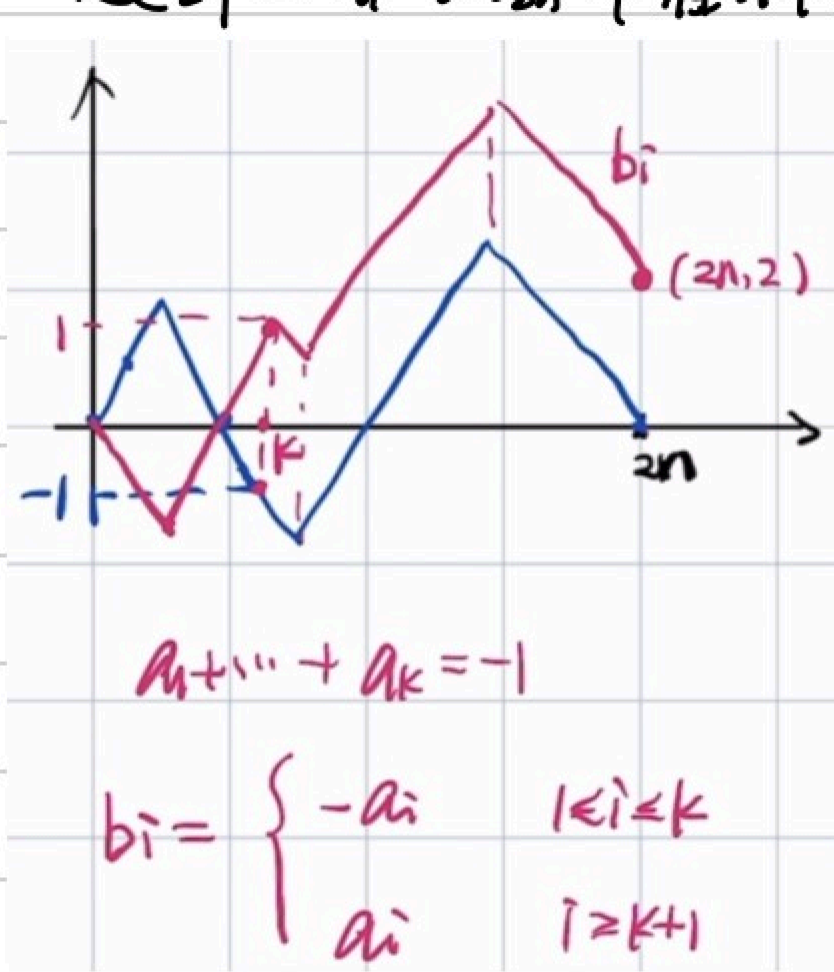
$$=\sum_{i=0}^{\infty}(1-e^{-x/2^i})=\sum_{i=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n!(2^i)^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!(1-2^{-n})}x^n$$

$f(x)=e^x g(x)$，其 x^n 项系数为 $\sum_{k=1}^{n}C_n^k\frac{1}{n!}\frac{(-1)^{k-1}}{1-2^{-k}}$

$\Longrightarrow e_n=\sum_{k=1}^{n}C_n^k\frac{(-1)^{k-1}}{1-2^{-k}}$

卡塔兰数　$\frac{1}{n+1}C_{2n}^n$

定义29：$a_1\sim a_{2n}$ 中有n个+1，n个-1，$\forall k\le 2n$ 有 $a_1+a_2+\cdots+a_k\ge 0$



红与蓝轨迹一一对应

红：n+1个+1，n-1个-1

考虑第一次达到 $y=-1$ 的情况（设为第k步）

令 $a_i=\begin{cases}-b_i, & 1\le i\le k\\ b_i & i\ge k+1\end{cases}$ ⇒ a_i 轨迹为蓝线

∴ 红、蓝轨迹一一对应

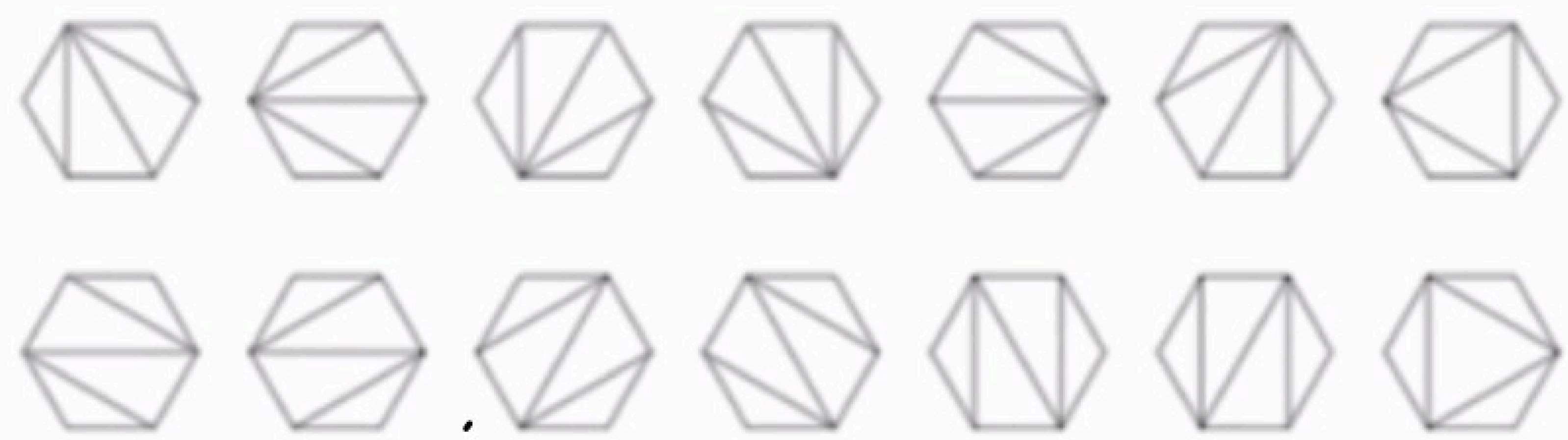
红：C_{2n}^{n+1} 条（选n+1条向上）

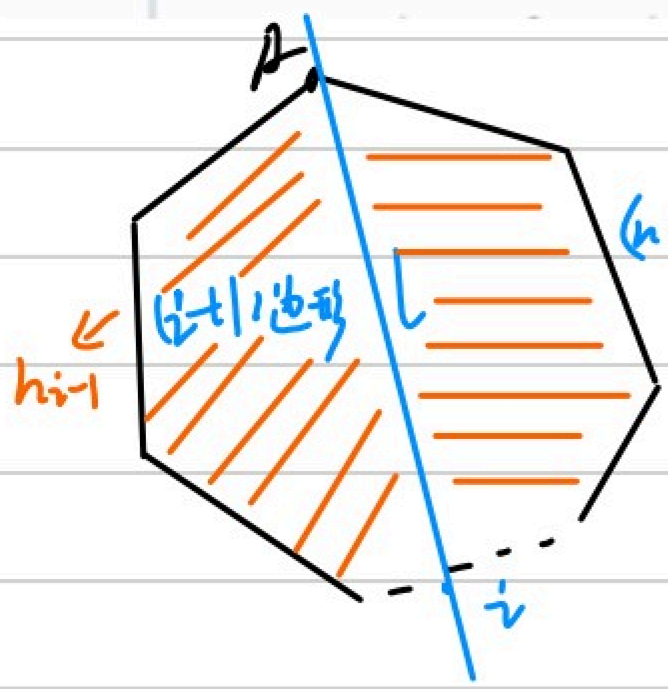
∴ 满条件的序列有：

$C_{2n}^n-C_{2n}^{n+1}=\frac{1}{n+1}C_{2n}^n$ ⇒ 卡塔兰数

定义30. n-1条对角线，分n+2边形，分成n个三角形，分法有 $\frac{1}{n+1}C_{2n}^n$ 种.（三角剖分个数）

$n=4$：分6边形. ⟶ 14种.





h_n 是 $n+2$ 边形的剖分数。

$$\boxed{h_n = \sum_{i=1}^{n-1} h_i h_{n-i}}$$ ⇒ 此式一定对应卡特兰数

考虑生成函数 $f(x) = \sum_{i=1}^{\infty} h_i x^i$

$$f(x) = h_1 x + h_1^2 x^2 + (h_1h_2+h_2h_1)x^3 + (h_1h_3+h_2h_2+h_3h_1)x^4+\cdots$$

$$= x + f^2(x) \Rightarrow f(x) = \frac{1\pm\sqrt{1-4x}}{2}$$

$$f(0)=0 \Rightarrow f(x) = \frac12 - \frac12\sqrt{1-4x} = \frac12-\frac12(1-4x)^{\frac12}$$

$$= \frac12 - \frac12\sum_{n=0}^{\infty} C_{\frac12}^{n} (-4x)^n$$ （广义二项式定理）

x^n 项系数：$C_{\frac12}^{n}(-4)^n\cdot\left(-\frac12\right) = (-1)^{n+1}2^{2n-1}\dfrac{\frac12(\frac12-1)(\frac12-2)\cdots(\frac12-n+1)}{n!}$

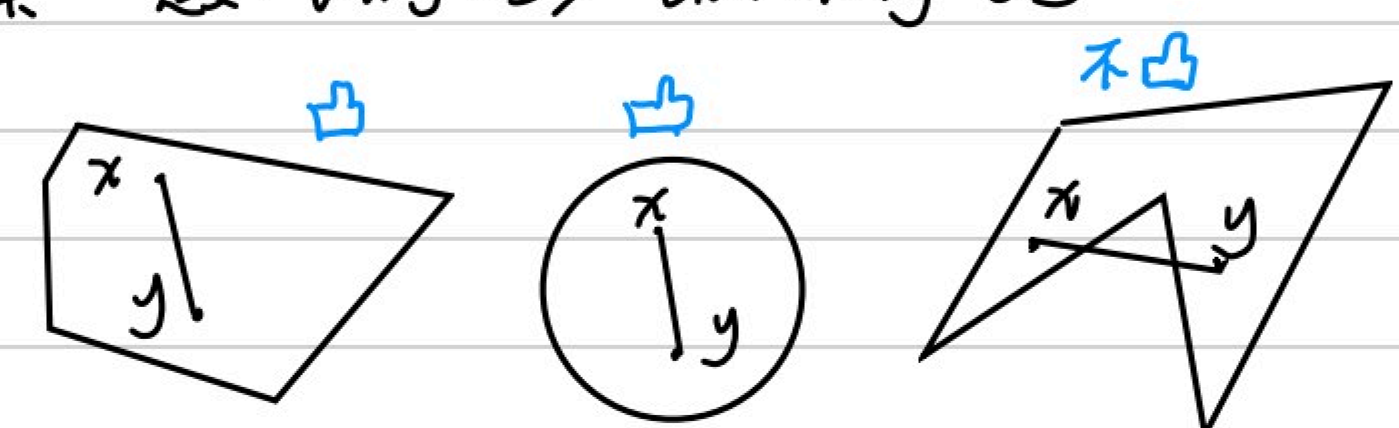
$$= (-1)^{n+1}2^{n-1}\frac{1}{n!}\left(1\cdot(1-2)\cdot(1-4)\cdots(1-2n+2)\right)$$

$$= \frac{(2n-3)!!}{n!}2^{n-1}$$

$$= \frac{(n-1)!\,2^{n-1}}{n!\,(n-1)!}(2n-3)!! = \frac{(2n-2)!}{n!\,(n-1)!} = \frac1n C_{2n-2}^{n-1}$$

组合几何

1. 凸集　定义：$\forall x,y\in S$，$tx+(1-t)y \in S$



性质：对于 $\lambda_1+\lambda_2+\cdots\lambda_k=1$，$\sum_{i=1}^{k}\lambda_i x_i \in S$（$x_n$为点，式子的意义为$x_1\cdots x_n$内部属于S）

证：对k归纳. $k=2$：凸集定义.

设k成立. 考虑$k+1$：

$$\sum_{i=1}^{k+1}\lambda_i X_i = \underline{(1-\lambda_{k+1})}\Big(\underbrace{\overbrace{\sum_{i=1}^{k}\frac{\lambda_i}{1-\lambda_{k+1}}X_i}^{Y_{k+1}}}_{归假\ \in S}\Big) + \lambda_{k+1}X_{k+1}$$

$$\triangleq (1-\lambda_{k+1})Y_{k+1}+\lambda_{k+1}X_{k+1} \in S$$

定义：点集X：凸包：$S=\{\sum_{i=1}^{k} t_i x_i \mid x_i\in X, \sum_{i=1}^{k} t_i=1, t_i\ge 0\}$

1° S为凸集（易验证）

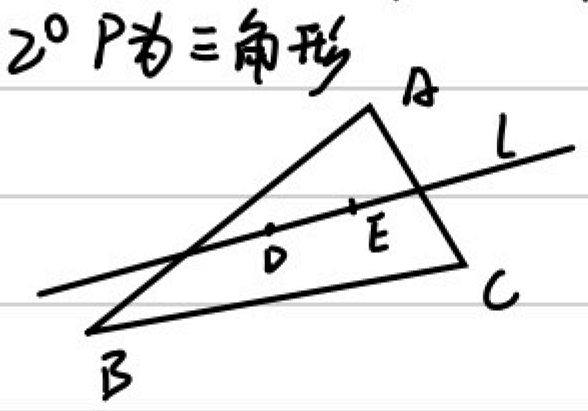
2° S是包括 x_i 的最小凸集

性质34：平面上不共面的5个点，一定存在4个点组成一个凸四边形

证明：按凸包形状讨论，设凸包为P

1° P为四边形/五边形，选凸包4个顶点

2° P为三角形



不妨设直线DE和BC不交，则$DECB$构成凸四边形

定理35、对于两个不交的凸集S_1、S_2，存在一条直线l将两者分开

证：引理：对闭凸集S，存在唯一一点$V\in S$，使$|OV|=\inf\{|OA| \mid A\in S\}$

取一列$\{A_n\}$满足$|OA_n|\to\inf\cdots$

则由于S凸 $\Rightarrow \frac{1}{2}(A_i+A_j)\in S$.

$\Rightarrow \frac{1}{4}|\overrightarrow{OA_j}+\overrightarrow{OA_i}|^2\ge \delta^2 \triangleq \inf\cdots$

$\Rightarrow |\overrightarrow{OA_j}-\overrightarrow{OA_i}|^2=2\underbrace{|\overrightarrow{OA_j}|^2}_{\to\delta^2}+2\underbrace{|\overrightarrow{OA_i}|^2}_{\to\delta^2}-\underbrace{|\overrightarrow{OA_j}+\overrightarrow{OA_i}|^2}_{\ge 4\delta^2}$

$\Rightarrow |\overrightarrow{OA_j}-\overrightarrow{OA_i}|^2\to 0$

又S闭 $\Rightarrow$ 存在 $V=\lim_{i\to\infty}\overrightarrow{A_i}$

将V代入田 $\Rightarrow$ V唯一

回到原定理，定义 $S=\{s_1-s_2 \mid s_1\in S_1,\ s_2\in S_2\}$

· 由引理，存在 $\bar{S}$（S闭包）中一点V到O距离最短，$\forall A\in S$，$t\in(0,1)$

$\Rightarrow |OV|^2\le|\overrightarrow{OV}+t(\overrightarrow{OA}-\overrightarrow{OV})|^2$

$=|\overrightarrow{OV}|^2+2t\,\overrightarrow{OV}\cdot(\overrightarrow{OA}-\overrightarrow{OV})+t^2|\overrightarrow{VA}|^2$

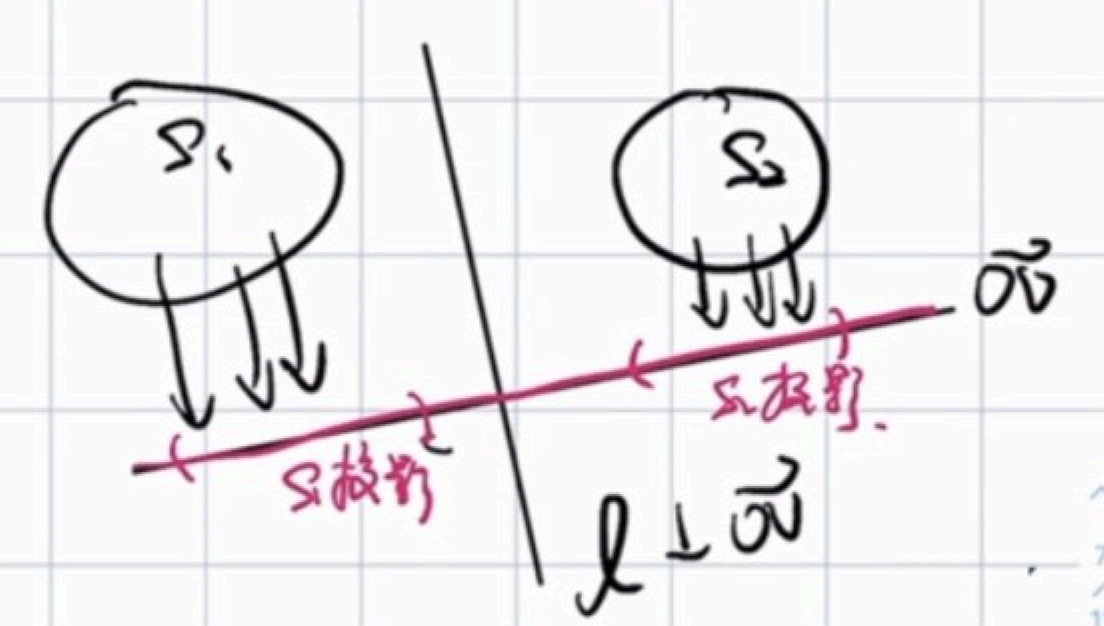
$\Rightarrow 0\le 2\overrightarrow{OV}\cdot\overrightarrow{OA}-2t|\overrightarrow{OV}|^2+t|\overrightarrow{VA}|^2$

令 $t\to 0$ 有 $\overrightarrow{OV}\cdot\overrightarrow{OA}\ge|OV|^2$

$\therefore \forall s_1\in S_1,\ s_2\in S_2$ 有

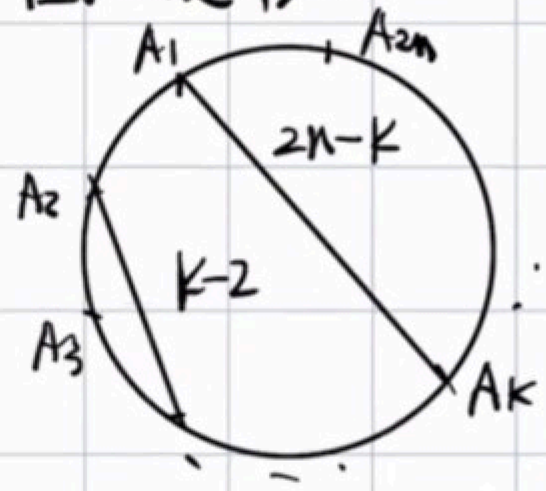
$\overrightarrow{OV}\cdot(\overrightarrow{OS_1}-\overrightarrow{OS_2})\ge|OV|^2$

$\Rightarrow \inf_{s_1\in S_1}\overrightarrow{OS_1}\cdot\overrightarrow{OV}\ge|OV|^2+\sup_{s_2\in S_2}\overrightarrow{OS_2}\cdot\overrightarrow{OV}$



题 36. 设 S 是平面上的 16 个点组成的集合, 记 $\chi(S)$ 为在以 S 中的点为端点画 8 条线段的画法总数, 满足没有任意两条线段都不相交, 也不共顶点. 对所有 S, 求 $\chi(S)$ 的最小可能值.

凸 2n 边形时.



考虑 A_1 与 A_K 相连.

令 e_n 是凸 2n 边形的画法个数. 则

$e_n = e_1e_{n-1} + e_2e_{n-2} + \cdots + e_{n-1}e_1$

卡塔兰数递推式

$\Rightarrow e_n = \frac{1}{n+1}C_{2n}^{n}$

考虑凸包上一点 P.

将余下的 $2n-1$ 个点按与 P 形成的角度标号, 形成 $A_1, A_2, \cdots A_{2n-1}$.

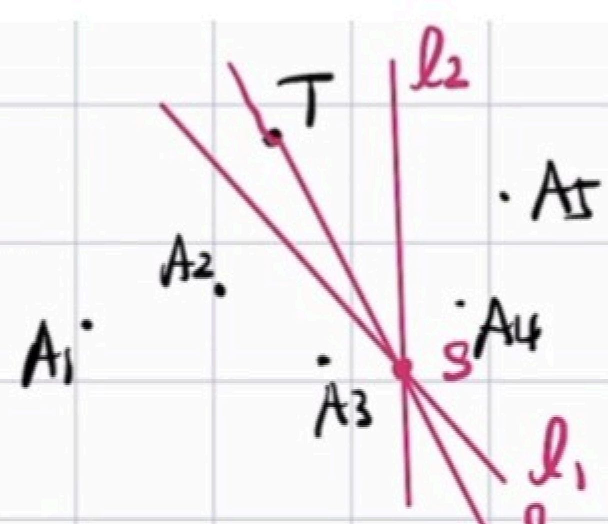
设 P 与 A_{2k+1} 相连. 将图形分为左右各 $2K$ 和 $2n-2K-2$ 个点.

设 C_n 是第 n 个卡特兰数.

$\chi(S) \geq C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0$ (归纳)

$= C_n$.

题 37. 在平面内, 有限点集的分割是指将点集分成两个不相交的子集 A, B, 使得存在一条直线不经过任意集合中的任意点且集合 A 中所有的点在直线的一侧, 集合 B 中的所有点在另一侧. 求平面内 n 个点的集合的分割个数的最大值.



l_1 和 l_2 对应 $\{A_1, \cdots A_n\}$ 的同一个分割

但 l_1、l_2 对应 $\{A_1, \cdots A_n, T\}$ 的不同的分割

考虑 $l_1 \cap l_2 = S$. 考虑直线 TS.

TS 将 $\{A_1, \cdots A_n\}$ 分为与 l_1, l_2 相同的分割. 故 l_1、l_2 将 $\{A_1 \cdots A_n\}$ 分为形如 $\{A_1, \cdots, A_k\}$ $\{A_{k+1}, \cdots A_n\}$ 的两部分.

$\Rightarrow$ 这样的分割总共有 n 个

$\Rightarrow a_{n+1} \leq n + a_n \leq \cdots$

$\leq n + (n-1) + \cdots + 1 + a_1$

$= C_{n+1}^{2} + 1$

$\Rightarrow a_n \leq C_n^2 + 1$.

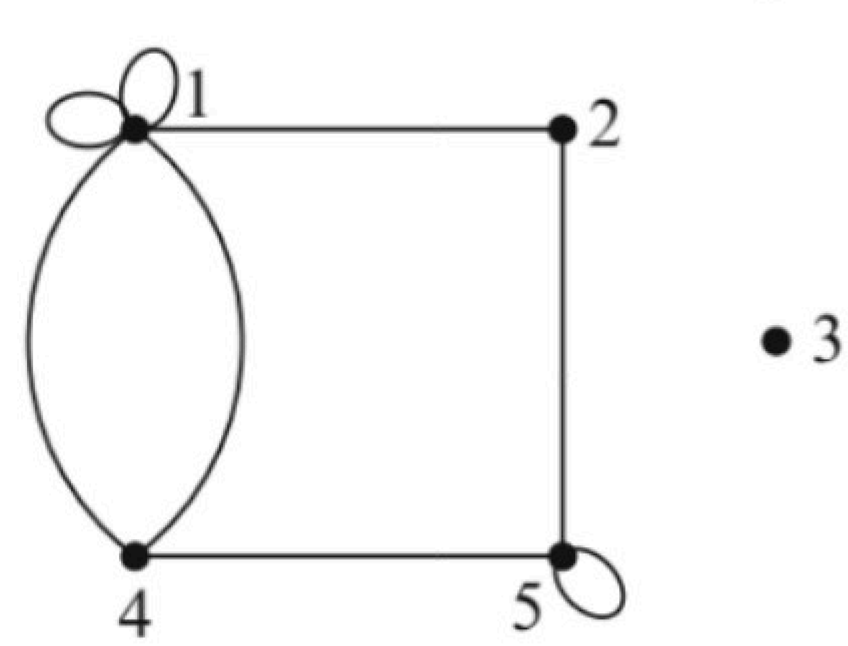
代数组合

Given a finite set S and integer $k \geq 0$, let $\binom{S}{k}$ denote the set of k-element subsets of S. A *multiset* may be regarded, somewhat informally, as a set with repeated elements, such as $\{1, 1, 3, 4, 4, 4, 6, 6\}$. We are only concerned with how many times each element occurs and not on any ordering of the elements. Thus for instance $\{2, 1, 2, 4, 1, 2\}$ and $\{1, 1, 2, 2, 2, 4\}$ are the same multiset: they each contain two 1's, three 2's, and one 4 (and no other elements). We say that a multiset M is *on* a set S if every element of M belongs to S. Thus the multiset in the example above is on the set $S = \{1, 3, 4, 6\}$ and also on any set containing S. Let $\left(\!\binom{S}{k}\!\right)$ denote the set of k-element multisets on S. For instance, if $S = \{1, 2, 3\}$ then (using abbreviated notation),

$$\binom{S}{2} = \{12, 13, 23\}, \quad \left(\!\!\binom{S}{2}\!\!\right) = \{11, 22, 33, 12, 13, 23\}.$$

We now define what is meant by a graph. Intuitively, graphs have vertices and edges, where each edge "connects" two vertices (which may be the same). It is possible for two different edges e and e' to connect the same two vertices. We want to be able to distinguish between these two edges, necessitating the following more precise definition. A (finite) *graph* G consists of a *vertex set* $V = \{v_1, \ldots, v_p\}$ and *edge set* $E = \{e_1, \ldots, e_q\}$, together with a function $\varphi\colon E \to \left(\!\binom{V}{2}\!\right)$. We think that if $\varphi(e) = uv$ (short for $\{u, v\}$), then e connects u and v or equivalently e is *incident* to u and v. If there is at least one edge incident to u and v then we say that the vertices u and v are *adjacent*. If $\varphi(e) = vv$, then we call e a *loop* at v. If several edges $e_1, \ldots, e_j$ $(j > 1)$ satisfy $\varphi(e_1) = \cdots = \varphi(e_j) = uv$, then we say that there is a *multiple edge* between u and v. A graph without loops or multiple edges is called *simple*. In this case we can think of E as just a subset of $\binom{V}{2}$ [why?]. ①

The *adjacency matrix* of the graph G is the $p \times p$ matrix $\boldsymbol{A} = \boldsymbol{A}(G)$, over the field of complex numbers, whose (i, j)-entry a_{ij} is equal to the number of edges incident to v_i and v_j. Thus $\boldsymbol{A}$ is a real symmetric matrix (and hence has real eigenvalues) whose trace is the number of loops in G. ② For instance, if G is the graph



① 无loop → 无aa形式元素
无mutiple edges
→ φ是双射
∴ $E \cong \left(\!\binom{V}{2}\!\right) = \binom{V}{2}$

② trace: $\sum a_{ii}$
∴ 由定义, 显然 a_{ii} 为第i个点的loop数, 则矩阵的迹为总loop数

then

$$A(G) = \begin{bmatrix} 2 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

A *walk* in G of *length* ℓ from vertex u to vertex v is a sequence $v_1, e_1, v_2, e_2, \ldots, v_\ell, e_\ell, v_{\ell+1}$ such that:

- Each v_i is a vertex of G.
- Each e_j is an edge of G.
- The vertices of e_i are v_i and v_{i+1}, for $1 \leq i \leq \ell$.
- $v_1 = u$ and $v_{\ell+1} = v$.

1.1 Theorem. *For any integer $\ell \geq 1$, the (i, j)-entry of the matrix $A(G)^\ell$ is equal to the number of walks from v_i to v_j in G of length ℓ.*

Proof. This is an immediate consequence of the definition of matrix multiplication. Let $A = (a_{ij})$. The (i, j)-entry of $A(G)^\ell$ is given by

$$(A(G)^\ell)_{ij} = \sum a_{ii_1} a_{i_1 i_2} \cdots a_{i_{\ell-1} j},$$

where the sum ranges over all sequences $(i_1, \ldots, i_{\ell-1})$ with $1 \leq i_k \leq p$. But since a_{rs} is the number of edges between v_r and v_s, it follows that the summand $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{\ell-1} j}$ in the above sum is just the number (which may be 0) of walks of length ℓ from v_i to v_j of the form

$$v_i, e_1, v_{i_1}, e_2, \ldots, v_{i_{\ell-1}}, e_\ell, v_j$$

(since there are a_{ii_1} choices for e_1, $a_{i_1 i_2}$ choices for e_2, etc.) Hence summing over all $(i_1, \ldots, i_{\ell-1})$ just gives the total number of walks of length ℓ from v_i to v_j, as desired. $\square$