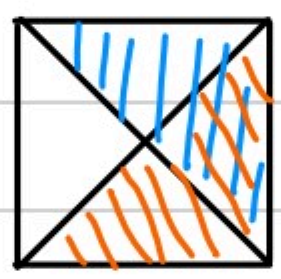
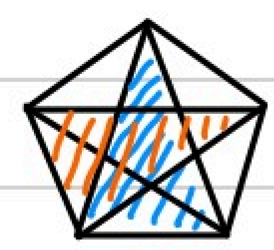
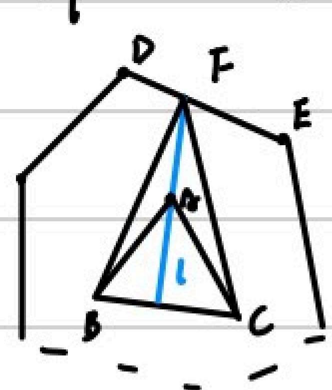
组合几何(2)

题 38. 已知平面上有 ~~2017~~ n 个点，且无三点共线. 称由这些点中任意三点组成的三角形面积最大的三角形为“好的”. 证明：不可能有多于 ~~2017~~ n 个好的三角形.

$n=4$:  $\Rightarrow$ 4个“好△”　$n=5$:  $\Rightarrow$ 5个“好△”

可推断为正n边形时有n个“好△”

解：1° 好△三个顶点都在凸包上，不妨设△ABC为好的，且A在凸包内部

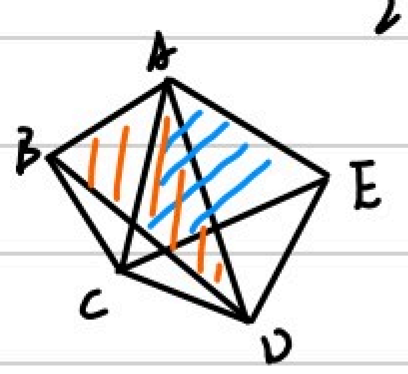


设△ABC中BC边上高交凸包于F，且F在DE上 $\Rightarrow S_{\triangle BCF} > S_{\triangle ABC}$

且 $\max\{S_{\triangle BCD}, S_{\triangle BCE}\} \geq S_{\triangle FBC} > S_{\triangle ABC}$　∴ △ABC不是好的

不妨设n个点的凸包为n多边形，1°中情况一定小于此时

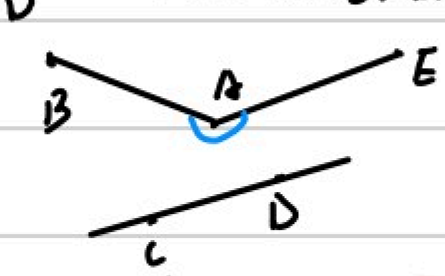
2° 当2个好△只有共顶点A时，好△只能为△ABD & △ACE 引理1



反证：若△ABC & △AED为好△

1) $S_{\triangle ABC} \geq S_{\triangle ABD}$　$d(C,AB) \geq d(D,AB) \Rightarrow$ 射线BA、CD相交

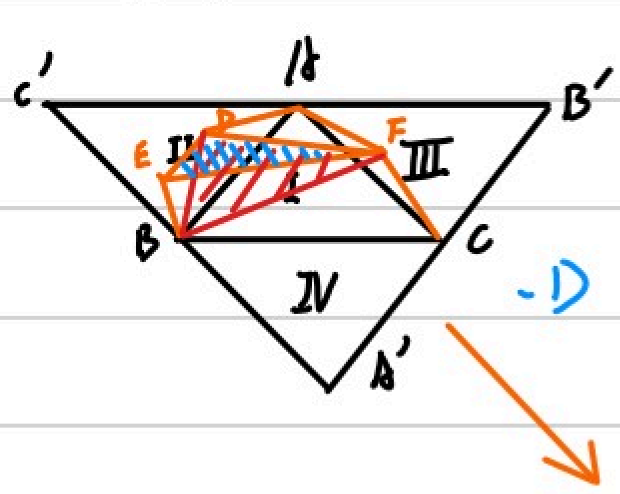
同理由 $S_{\triangle AED} \geq S_{\triangle ACE} \Rightarrow$ EA、DC也相交



$\Rightarrow \angle BAE > 180^\circ$

∴ 与凸性矛盾

3° 两个好△不共顶点　引理2



△DEF为好△ $\Rightarrow$ D、E、F分别落于II、III、IV中

1) D、E、F在△A′B′C′中

否则不妨设D在A′B′异侧 $\Rightarrow S_{\triangle ABD} > S_{\triangle ABC}$，矛盾

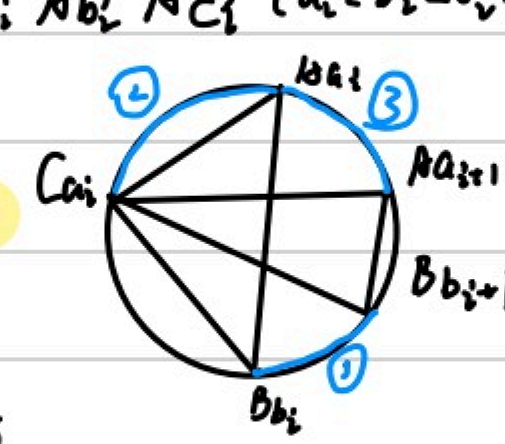
2) D、E、F在凸包上 $\Rightarrow$ D、E、F不在△ABC内部

3) DE都在区域II中，不妨设F在III中

∵ E在A′C′右侧 $\Rightarrow S_{\triangle DEF} < S_{\triangle ABC}$，与△DEF为好△矛盾

回到原题，设凸n边形n个点顺次为 $A_1, A_2, \cdots, A_n$，设好△为 $\triangle A_{a_i}A_{b_i}A_{c_i}$ $(a_i < b_i < c_i)$

再将△按 (a_i, b_i, c_i) 字典序排列　$a_1 \leq a_2 \leq \cdots a_m$



下面证明 $a_1 \leq a_2 \leq \cdots \leq a_m \leq b_1 \leq b_2 \leq \cdots \leq b_m \leq c_1 \leq c_2 \leq \cdots \leq c_m$

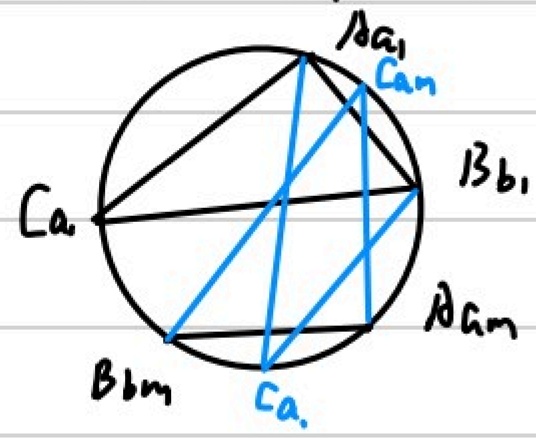
1) $b_1 \leq b_2 \leq \cdots \leq b_m$

否则存在 $b_i > b_{i+1} \Rightarrow a_i < a_{i+1}$　若 $C_{a_{i+1}}$ 出现在①或②，与引理2矛盾

若 $C_{a_{i+1}} = C_{a_i}$，与引理2矛盾

2) $a \leq a_2 \leq \cdots \leq c_m$　与 b 相似

3) $a_m \le b_1$, $b_m \le c_1$



若 $a_m > b_1$，则有右图

而 Ca_m 不可能在 Aa_1 右侧，矛盾！

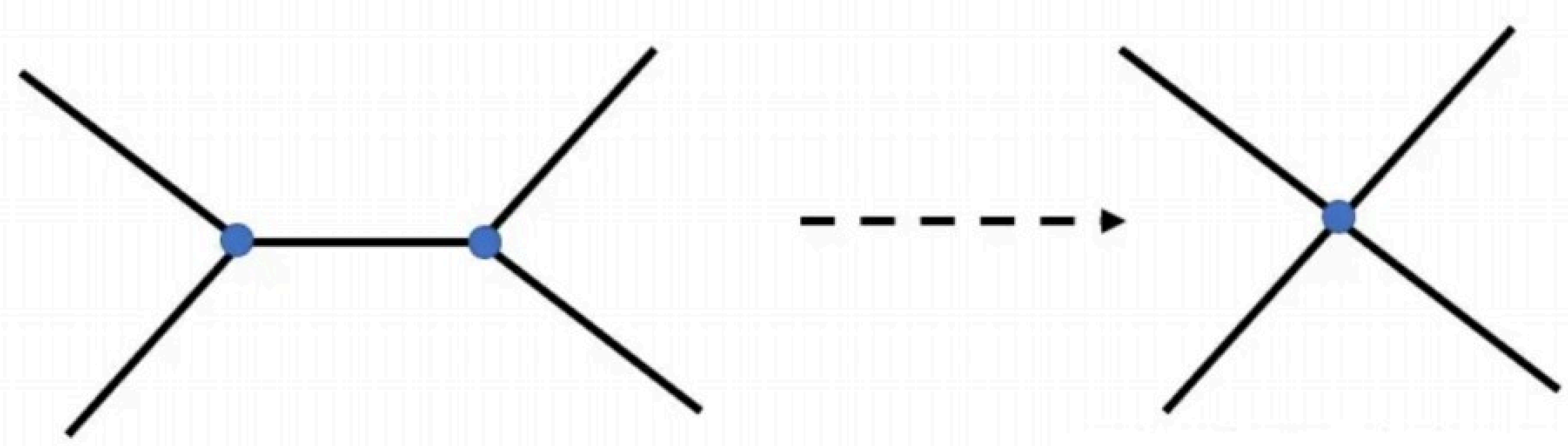
$S_i = a_i + b_i + c_i$ 是单调↑的

$S_m - S_i = c_n + (b_m - c_1) + (a_m - b_1) - a_n \le n + 0 + 0 + 0 = n$

$\Rightarrow$ 最多 n 个好△

欧拉公式：对于凸多面体，有 $V-E+F=2$（V为顶点，E为棱，F为面）

证：对顶点数 n 采用数学归纳法. 假设 $v=k$ 时凸多面体满足 $v-e+f=2$。对于一个顶点数为$v'=(k+1)$、边数为 e'、面数为 f' 的凸多面体的一条边进行“收缩”——把一条边的两个顶点合并成一个顶点，如下图所示



于是这个多面体的顶点数变为 $v'-1=k=v$,边数变为 $e'-1=e$，面数 $f'=f$.根据 $v-e+f=2$ 可知，

$$(v'-1)-(e'-1)+f'=v'-e'+f'=2 \quad .\square$$

推论40.（求所有正多面体：各面为全等的正多边形，各多面角全等）

设每个面为正 a 边形，每个顶点连 b 条棱

则：$aF = 2E = bV \triangleq k$

$\Rightarrow F=\frac{k}{a}$, $E=\frac{k}{2}$, $V=\frac{k}{b}$

代入欧拉公式：

$\frac{k}{a}-\frac{k}{2}+\frac{k}{b}=2=\frac{k}{E}$

$\Rightarrow \frac{1}{a}-\frac{1}{2}+\frac{1}{b}=\frac{1}{E}$

$\frac{1}{a}-\frac{1}{2}+\frac{1}{b}>0 \Rightarrow 3\le a,b<6$

代入验证：

$a=b=3$: $\Rightarrow E=6$ ……

解得：$(F,E,V)=(4,6,4)$, $(8,12,6)$,

$(20,30,12)$, $(12,30,20)$

$(6,12,8)$

拓展：欧拉示性数

对于多面体，$\chi=V-E+F$。其为拓扑不变量，即在同胚意义下相同的图形，χ 不变

（凹多面体与凸多面体不同胚，示数不为2；球与凸多面体同胚，示性数也为2）

§67.1 Degrees

Prototypical example for this section: $z \mapsto z^d$ *has degree* d.

For any $n > 0$ and map $f : S^n \to S^n$, consider

$$f_* : \underbrace{H_n(S^n)}_{\cong \mathbb{Z}} \to \underbrace{H_n(S^n)}_{\cong \mathbb{Z}}$$

which must be multiplication by some constant d. This d is called the **degree** of f, denoted $\deg f$.

> **Question 67.1.1.** Show that $\deg(f \circ g) = \deg(f)\deg(g)$.

Note that we haven't defined $H_n(S)$. in fact it's the nth homology group of S.

Definition 64.2.3. The n**th homology group** $H_n(X)$ is defined as

$$H_n(X) := Z_n(X)/B_n(X).$$

quotient group

Ranhou wo you zhaodao "shangqun" de dingyi, [illegible]

Definition 3.3.5. A subgroup N of G is called **normal** if it is the kernel of some homomorphism. We write this as $N \trianglelefteq G$.

Definition 3.3.6. Let $N \trianglelefteq G$. Then the **quotient group**, denoted G/N (and read "G mod N"), is the group defined as follows.

- The elements of G/N will be the left cosets of N.
- We want to define the product of two cosets C_1 and C_2 in G/N. Recall that the cosets are in bijection with elements of Q. So let q_1 be the value associated to the coset C_1, and q_2 the one for C_2. Then we can take the product to be the coset corresponding to $q_1 q_2$.

  Quite importantly, **we can also do this in terms of representatives of the cosets**. Let $g_1 \in C_1$ and $g_2 \in C_2$, so $C_1 = g_1 N$ and $C_2 = g_2 N$. Then $C_1 \cdot C_2$ should be the coset which contains $g_1 g_2$. This is the same as the above definition since $\phi(g_1 g_2) = \phi(g_1)\phi(g_2) = q_1 q_2$; all we've done is define the product in terms of elements of G, rather than values in H.

  Using the gN notation, and with Remark 3.3.3 in mind, we can write this even more succinctly:

$$(g_1 N) \cdot (g_2 N) := (g_1 g_2) N.$$

And now you know why the integers modulo n are often written $\mathbb{Z}/n\mathbb{Z}$!

[illegible] homology group [illegible]

话回来，我之前看 attaching word 竟然没看懂，真是傻了。就是按图延某个顺序写出来。

(见N-20221229)