Differential Forms and Stokes: The Geometry of Boundary Measurements

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§1.1 A theorem looking for its natural language

The fundamental theorem of calculus says

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Green's theorem, the divergence theorem, and Stokes' theorem look more complicated, but they share the same shape: integrate a derivative over a region, and the answer lives on the boundary.

Differential forms are the language in which this pattern becomes one theorem:

$$\int_M d\omega = \int_{\partial M} \omega.$$

The purpose of this chapter is not to glorify notation. It is to make clear why dx, dy, pullback, orientation, and boundary signs belong together.

§1.2 One-forms: measuring motion

A tangent vector tells us a direction of motion. A covector measures such a direction. At a point p, a covector is a linear map

$$\alpha : T_pM \to \mathbb{R}.$$

A 1-form is a smoothly varying covector.

In the plane, a typical 1-form is

$$\omega = P(x,y)\,dx + Q(x,y)\,dy.$$

If

$$v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y},$$

then

$$\omega(v) = Pa + Qb.$$

Thus a 1-form is not a tiny arrow. It is a measuring rule for arrows.

§1.3 The pullback experiment

Let $\gamma(t) = (x(t), y(t))$ be a parametrized curve. Pulling back the form

$$\omega = P(x,y)\,dx + Q(x,y)\,dy$$

to the interval gives

$$\gamma^*\omega = \big(P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\big)\, dt.$$

Therefore

$$\int_\gamma \omega = \int_a^b \big(P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\big)\, dt.$$

This is the usual line integral. The conceptual advantage is that the curve integral is reduced to ordinary integration after pullback.

§1.4 Two-forms: measuring oriented area

The wedge product is antisymmetric:

$$dx \wedge dy = -dy \wedge dx, \qquad dx \wedge dx = 0.$$

A 2-form such as

$$f(x, y)\, dx \wedge dy$$

measures signed area with density f. The sign changes if the orientation is reversed.

This is why forms are naturally tied to oriented geometry. A form does not merely ask "how large?" It asks "how large, with which orientation?"

§1.5 Exterior derivative as one operator

For a function f,

$$df = f_x\, dx + f_y\, dy.$$

For a 1-form $\omega = P\, dx + Q\, dy$,

$$d\omega = (Q_x - P_y)\, dx \wedge dy.$$

The rule

$$d^2 = 0$$

is the compact source of several familiar identities: gradients have zero curl, curls have zero divergence, and exact forms are closed.

§1.6 A familiar theorem reappears

Apply Stokes to a planar region D and a 1-form $\omega = P\, dx + Q\, dy$:

$$\int_D d\omega = \int_{\partial D} \omega.$$

Writing this in coordinates gives

$$\iint\limits_D (Q_x - P_y)\, dx\, dy = \int_{\partial D} P\, dx + Q\, dy.$$

This is Green's theorem. The point is not that Green's theorem becomes shorter, but that its boundary nature becomes unavoidable.

§1.7 Orientation is not decoration

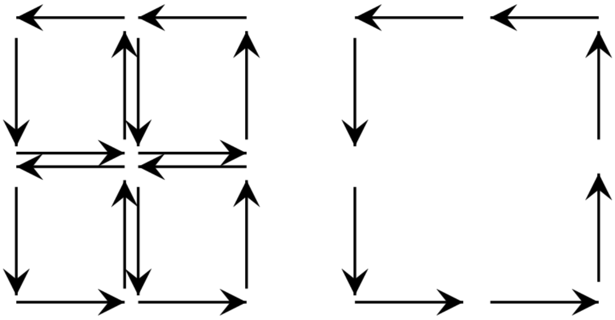

Figure 1.1: The orientation of a patch forces an induced orientation on its boundary.

The boundary of an oriented region inherits an orientation. If this convention is changed, Stokes' theorem changes sign. So orientation is not a drawing preference; it is part of the data needed for the theorem to be true.

§1.8 Why arc length is a different kind of geometry

It is tempting to treat ds as a 1-form, but arc length is not linear in the velocity:

$$ds(\gamma') = \|\gamma'\|\,dt.$$

A 1-form is linear in the tangent vector. A norm is not. This distinction separates two kinds of geometry:

differential forms	oriented measurement and integration
Riemannian metrics	lengths, angles, and distances

Both are important, but they should not be confused.

§1.9 Closed forms can detect holes

The distinction between closed and exact forms is one of the first places where analysis notices topology. On $\mathbb{R}^2 \setminus \{0\}$, consider

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

One computes that

$$d\omega = 0.$$

So ω is closed. But it is not exact on $\mathbb{R}^2 \setminus \{0\}$. Indeed, along the unit circle $\gamma(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$, we get

$$\gamma^*\omega = dt,$$

and therefore

$$\int_\gamma \omega = 2\pi.$$

If $\omega = df$, the integral around a closed loop would be 0. The nonzero integral detects the missing origin.

This is a beautiful lesson: a differential form can carry topological information.

§1.10 A table of classical shadows

The same theorem appears in several costumes:

dimension	form	classical theorem
1	f	$\int_a^b df = f(b) - f(a)$
2	$P\,dx + Q\,dy$	Green's theorem
3	flux form	divergence theorem
3	circulation form	classical Stokes theorem

The table is not meant as a replacement for calculation. It is meant to prevent the common mistake of treating these theorems as unrelated facts.

§1.11 Flux as a form

In $\mathbb{R}^3$, a vector field

$$\mathbf{F} = (P, Q, R)$$

corresponds to the 2-form

$$\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

Then

$$d\omega = (P_x + Q_y + R_z)\,dx \wedge dy \wedge dz.$$

Thus the divergence theorem is exactly Stokes' theorem for this 2-form:

$$\int_M d\omega = \int_{\partial M} \omega.$$

This is a good example of how vector calculus becomes cleaner when translated into forms.

§1.12 The moral of pullback

Integration should not depend on a lucky parametrization. If a surface is parametrized by

$$\Phi : U \to M,$$

then an integral over M is computed by pulling the form back to U:

$$\int_M \omega = \int_U \Phi^* \omega.$$

The pullback automatically inserts the Jacobian-like factors and orientation signs. This is why forms are the natural objects to integrate: they know how to move backward along maps.

§1.13 Exit line

Differential forms are a language for quantities that can be pulled back and integrated. Stokes' theorem says that the integral of a derivative over a body is a boundary measurement.

The short memory is:

forms measure; pullbacks transport;
d differentiates;
Stokes moves to the boundary.