A Nontrivial Loop at Zero: FinSet K-Theory and Stable Homotopy

N-20251217

§1.1 Why this second note exists

This note is the reflective version of the earlier summer memo. The first reaction to the line

$$0 = 1 - 1 = -1 + 1 = 0$$

was not the answer eventually given by the lecture. A few days before writing this correction, while I was organizing my homepage, my roommate Su Yuyang saw the same line. His immediate thought was the familiar analytic story around

$$\sum_{k \geq 0} (-1)^k = 1 - 1 + 1 - 1 + \cdots,$$

whose Cesàro value and Abel value are $1/2$, and which is also connected with the analytic continuation of the Dirichlet eta function. This was also my own first thought. The string $1 - 1 = -1 + 1$ looks like a rearrangement or regularization trick before it looks like topology.

But the meme and the lecture poster clearly featured the Hopf fibration and K-theory. That made the analytic-summation explanation feel too small. Following the same original line of thought, I then searched for a K-theoretic or categorical mechanism in which an apparently impossible cancellation becomes legitimate. This led naturally to the Eilenberg swindle.

The model I had in mind was this. If an object admits an infinite decomposition

$$S = A \oplus A \oplus A \oplus \cdots,$$

then adding or removing one copy of A may not change S. Formally, one shifts the infinite direct sum:

$$S \cong A \oplus S.$$

In a Grothendieck group this kind of absorption can force

$$[A] = 0.$$

In the language of the meme, I was reading

$$0 = 1 - 1 = -1 + 1 = 0$$

as a symbolic version of infinite stability: by regrouping an infinite alternating expression, one can make the same expression look like 0, 1, or a vanishing object. This was my "Level 6" interpretation: the ultimate trick behind the line is that infinity absorbs finite differences, and therefore certain K-groups or invariants vanish.

That guess was not completely unrelated. It correctly noticed that K-theory is about addition, cancellation, group completion, and stability. It also correctly noticed that an equality such as $0 = 0$ might be hiding a process rather than an elementary arithmetic fact. But the direction was almost the opposite of Prof. Peng's point. My first interpretation emphasized:

infinite objects,
analytic or algebraic cancellation,
triviality and vanishing.

The lecture's point is instead:

finite sets,
homotopy of a finite creation–swap–annihilation process,
nontriviality of a loop.

So the slogan of this note is:

the important object is not the endpoint 0, but the loop based at 0.

The loop is created by making a $+$ and a $-$, swapping them, and annihilating them again:

$$\varnothing \sim \{+,-\} \sim \{-,+\} \sim \varnothing.$$

Under the K-theory of finite sets, this loop becomes a nontrivial element. Under the Barratt–Priddy–Quillen theorem, it corresponds to the stable Hopf class.

§1.2 The lecture starts from ordinary equations

The slide deck begins with familiar equations: Pythagoras, Euler's identity, Stokes' theorem, the Navier–Stokes equation, the Langlands slogan, and then the strange meme equation. This is pedagogically important. It suggests that an equation is not merely a string of symbols with a truth value. An equation may also encode a method, a structure, or a path.

In elementary arithmetic,

$$0 = 1 - 1 = -1 + 1 = 0$$

is boring. Both sides are zero. But the lecture asks us to stop looking only at the value. In a higher structure, there can be more than one way for an object to equal itself. The identity loop at 0 and the creation–swap–annihilation loop at 0 need not be the same.

> **Remark 1.2.1** — This is the conceptual bridge to homotopy type theory and higher category theory. A statement $a = b$ can be represented by a path from a to b. Then two proofs of equality may themselves be compared by a homotopy. The lecture uses this philosophy in a concrete topological model rather than as formal type theory.

§1.3 The Hopf fibration as the first nontrivial map

The first mathematical anchor is the Hopf fibration. Define

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

The Hopf map is

$$H : S^3 \to \mathbb{CP}^1 \cong S^2, \qquad (z_1, z_2) \mapsto [z_1 : z_2].$$

For every point of S^2, its preimage is a circle. These circles are linked in a highly organized way.

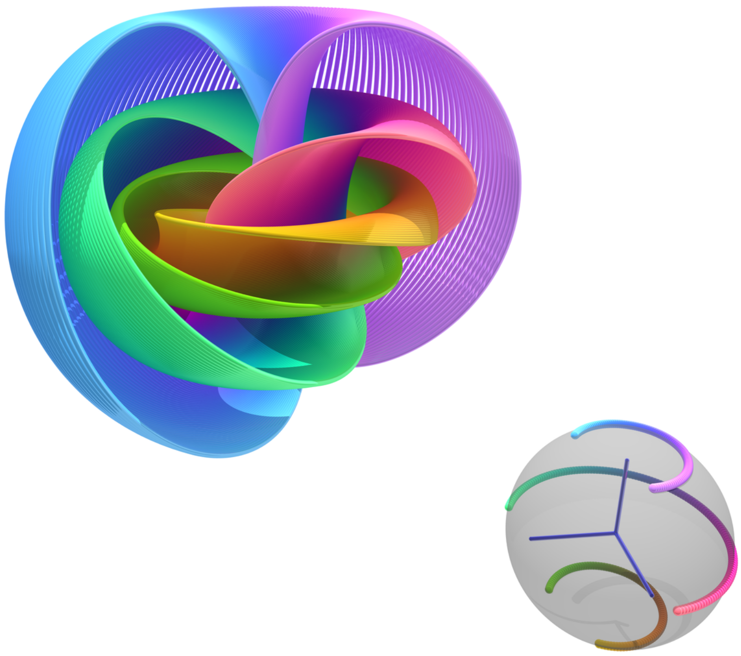

Figure 1.1: the Hopf fibration visualized by linked fibres.

The Hopf fibration matters because it is not null-homotopic. It represents a nontrivial element of

$$\pi_3(S^2).$$

In beginner language: even though S^3 and S^2 are both spheres, the map $H : S^3 \to S^2$ cannot be continuously shrunk to a constant map.

Definition 1.3.1 (Null-homotopic map)**.** A map $f : X \to Y$ is null-homotopic if it is homotopic to a constant map. Equivalently, there is a continuous family of maps $f_t : X \to Y$ with $f_0 = f$ and f_1 constant.

The Hopf map is the first warning: a process can be topologically nontrivial even if it looks, from far away, like a map between simple spaces.

§1.4 Homotopy groups and the first stable class

For a based space X, the group $\pi_n(X)$ is made from based maps $S^n \to X$, modulo based homotopy. The lecture reviews the low-dimensional intuition:

$$\pi_0(X) = \text{connected components}, \qquad \pi_1(X) = \text{loops modulo homotopy}.$$

Higher homotopy groups generalize this to maps from higher-dimensional spheres.

For spheres, the groups are easy below the diagonal:

$$\pi_i(S^n) = 0 \qquad (i < n).$$

On the diagonal, one has the degree:

$$\pi_n(S^n) \cong \mathbb{Z}.$$

But above the diagonal the world becomes subtle. The Hopf map gives the famous example

$$\pi_3(S^2) \neq 0.$$

The stable homotopy groups of spheres are obtained by repeatedly suspending:

$$\pi_i^s(\mathbb{S}) = \lim_{n\to\infty} \pi_{i+n}(S^n).$$

The Hopf map contributes the first stable nontrivial class

$$\eta \in \pi_1^s(\mathbb{S}) \cong \mathbb{Z}/2.$$

This class has order two: doing it twice becomes trivial, but doing it once is not trivial.

§1.5 From natural numbers to finite sets

The lecture then changes viewpoint. Instead of defining the natural numbers as primitive symbols, define them as isomorphism classes of finite sets:

$$\mathbb{N} \cong \pi_0(\mathrm{FinSet}).$$

Here a finite set of size n represents the number n. Disjoint union represents addition:

$$A + B := A \sqcup B.$$

To get the integers, one takes a group completion. Informally, one allows formal differences

$$A - B.$$

Thus the number $1 - 1$ is represented by a pair consisting of one positive point and one negative point.

> **Remark 1.5.1 —** The corresponding slide phrases this as a move from numbers to finite sets: the natural number n is represented by an n-element set, and addition is represented by disjoint union. This is the first categorification step in the lecture: instead of remembering only the cardinality, one keeps the finite set itself and later also its symmetries.

Definition 1.5.2 (Grothendieck group)**.** Given a commutative monoid M, its Grothendieck group $K_0(M)$ is the universal abelian group receiving a monoid homomorphism from M. For finite sets under disjoint union this gives

$$K_0(\mathrm{FinSet}) \cong \mathbb{Z}.$$

So far, this still only explains the value of $1 - 1$. It does not yet explain why the path

$$0 \to 1 - 1 \to -1 + 1 \to 0$$

can be interesting. For that, we must remember not just the set of isomorphism classes, but the space of finite sets and isomorphisms.

§1.6 The space of finite sets

The slide deck asks us to consider the moduli space

$$\mathcal{F} = \mathrm{FinSet}/\sim.$$

This is not merely a set of cardinalities. It remembers automorphisms of finite sets. A finite set with n elements has automorphism group S_n, the symmetric group.

Remark 1.6.1 — The slide at this point depicts the moduli space of finite sets as a space whose n-point component remembers S_n. In notebook language, the important formula is

$$\mathcal{F} \simeq \coprod_{n\geq 0} BS_n.$$

Thus the component remembers the number n, while loops in that component remember permutations.

One can think of $\mathcal{F}$ as a disjoint union of classifying spaces:

$$\mathcal{F} \simeq \coprod_{n\geq 0} BS_n.$$

This formula is the first place where the hidden loops appear. A point in BS_n remembers a finite set of size n, but loops at that point remember permutations of the set.

Thus the finite set with one positive and one negative point is not just a cardinality-zero object. It can have a nontrivial path coming from swapping the two labelled positions before annihilation.

§1.7 The K-space of finite signed sets

To group-complete finite sets at the level of spaces, the slides introduce a space $K(\mathcal{F})$. Its points are finite ordered configurations whose elements are marked by $+$ or $-$. Adjacent opposite signs may cancel.

Remark 1.7.1 — The slide introduces the K-space by replacing finite sets with signed finite configurations. The picture is simple enough to state in words: points marked $+$ and points marked $-$ may be created and cancelled in opposite-signed pairs, and the empty configuration is used as the base point.

The empty configuration $\varnothing$ is the base point. The configuration $\{+,-\}$ has total signed cardinality zero, but it is not identical to the empty configuration before one performs cancellation. This distinction is the seed of the whole example.

The higher K-groups are defined by

$$K_i(\mathrm{FinSet}) = \pi_i(K(\mathcal{F}), \varnothing).$$

Thus K_0 records components, while K_1 records loops based at the empty configuration.

§1.8 The loop h

Now the lecture constructs the loop

$$h : \varnothing \sim \{+,-\} \sim \{-,+\} \sim \varnothing.$$

It has three stages.

First, create a cancelling pair:

$$\varnothing \longrightarrow \{+,-\}.$$

Second, swap the two points:

$$\{+,-\} \longrightarrow \{-,+\}.$$

Third, annihilate the cancelling pair:

$$\{-,+\} \longrightarrow \varnothing.$$

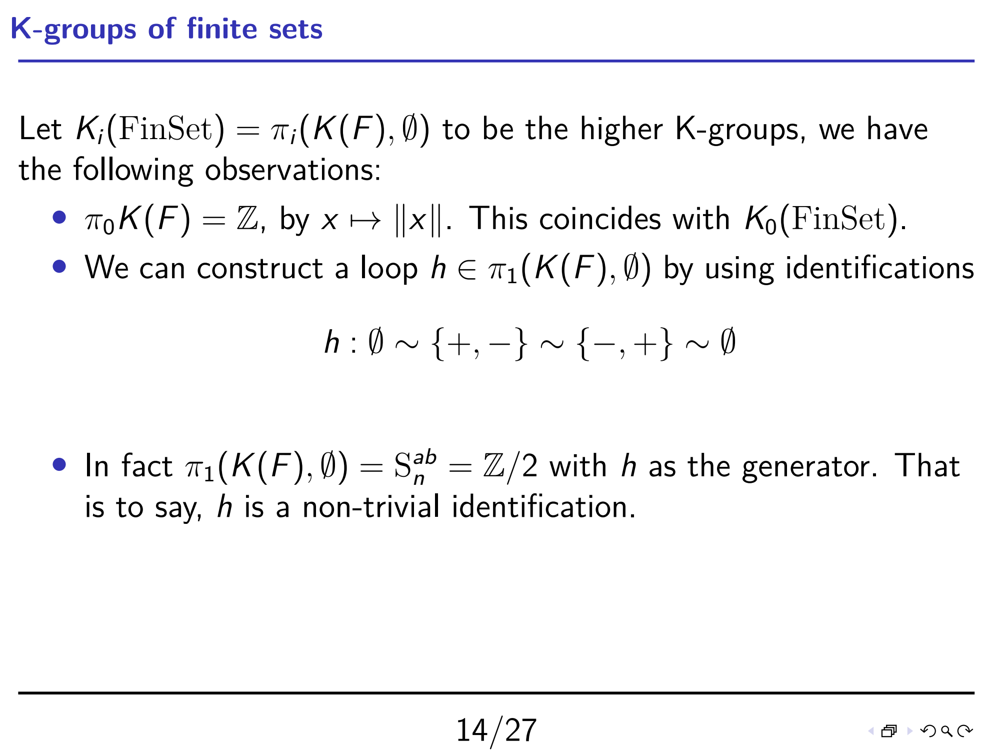

Figure 1.2: From the lecture slides: the loop h represented by creation, swap, and annihilation.

This is the corrected meaning of

$$0 = 1 - 1 = -1 + 1 = 0.$$

It is not the claim that zero equals zero. It is the claim that there is a particular loop at zero. The middle step, the swap, is the nontrivial part.

Remark 1.8.1 — If the two points were not moved, the movie would look like a boring creation followed by immediate annihilation. The swap forces a braid-like motion. In configuration space, this motion wraps around the collision locus where the two points would meet.

The slides record the key computation:

$$K_1(\mathrm{FinSet}) \cong \mathbb{Z}/2,$$

and the loop h gives the nontrivial element. Doing the swap twice is null-homotopic, but doing it once is not.

§1.9 Why the first guess was close but reversed

The analytic reading begins with the infinite alternating expression

$$1 - 1 + 1 - 1 + \cdots.$$

Cesàro summation replaces the raw partial sums

$$1,\ 0,\ 1,\ 0,\dots$$

by their averages, which tend to 1/2. Abel summation studies

$$1 - r + r^2 - r^3 + \cdots = \frac{1}{1+r}$$

and then lets $r \to 1^-$, again obtaining 1/2. Analytic continuation of the Dirichlet eta function gives another sophisticated version of the same cultural memory.

This is why Su Yuyang's first reaction and my first reaction were natural. The line

$$0 = 1 - 1 = -1 + 1 = 0$$

looks as if it is inviting a discussion of how formal rearrangements, limiting processes, and regularizations change the meaning of an alternating expression.

The Eilenberg swindle pushes that same instinct from analysis into algebra. Suppose a category admits countable direct sums, and an object S satisfies

$$S \cong A \oplus S.$$

Then in a Grothendieck group one obtains

$$[S] = [A] + [S],$$

so formally

$$[A] = 0.$$

This is the mechanism behind many vanishing arguments: an infinite object absorbs finite additions, and the invariant becomes trivial. In my first interpretation, the meme equation was a compressed version of this phenomenon. The "truth" was supposed to be that infinity makes cancellation too flexible, so the sophisticated K-theoretic machinery collapses into an infinite arithmetic trick.

The actual lecture keeps several words from that interpretation but reverses their role. It still cares about K-theory, addition, cancellation, and stability. But it does not use infinity to erase a finite object. It uses finite configurations to produce a nontrivial loop. The contrast is:

first interpretation	lecture's mechanism
infinite alternating series	finite signed configuration
Cesàro/Abel/eta regularization	creation–swap–annihilation path
Eilenberg swindle	$K(\mathrm{FinSet})$
absorption at infinity	nontrivial loop at the base point
vanishing of invariants	generator of $K_1(\mathrm{FinSet}) \cong \mathbb{Z}/2$
triviality	stable Hopf nontriviality

Thus the first guess was close in vocabulary but opposite in direction. It looked for the triviality of the infinite. Prof. Peng's point is the nontriviality of the finite.

§1.10 Finite sets and stable spheres

The surprising theorem behind the example is the Barratt–Priddy–Quillen theorem. In the form used by the lecture, it says that the K-theory of finite sets is the sphere spectrum:

$$K_i(\mathrm{FinSet}) \cong \pi_i^s(\mathbb{S}).$$

Theorem 1.10.1 (Barratt–Priddy–Quillen, lecture form)
The K-theory of finite sets is the sphere spectrum. Equivalently,

$$K_i(\mathrm{FinSet}) \cong \pi_i^s(\mathbb{S}).$$

In the slide deck this theorem is presented as the conceptual bridge: a finite-set loop can detect a stable homotopy class of spheres.

This theorem explains why the finite-set loop h can be compared to the Hopf class. Under BPQ, the nontrivial element of

$$K_1(\mathrm{FinSet}) \cong \mathbb{Z}/2$$

corresponds to

$$\eta \in \pi_1^s(\mathbb{S}) \cong \mathbb{Z}/2.$$

This is the stable Hopf class.

Another way to say this is that $K(\mathcal{F})$ can be seen as a stable configuration space of framed points:

$$K(\mathcal{F}) \simeq \mathrm{Conf}^{\mathrm{fr}}(\mathbb{R}^\infty).$$

The creation–swap–annihilation loop becomes a framed one-dimensional motion. Its framing carries the $\mathbb{Z}/2$-twist.

§1.11 Relation with the classical Hopf map

The classical Hopf fibration represents a nontrivial element of $\pi_3(S^2)$. Stabilizing the Hopf map gives the stable element

$$\eta \in \pi_1^s(\mathbb{S}).$$

The finite-set loop h gives the same class under the BPQ theorem.

Thus the meme ladder is not merely free association. The chain is:

$$\begin{gathered}
\text{Hopf fibration } S^3 \to S^2 \\
\Downarrow \\
\text{stable Hopf element } \eta \in \pi_1^s(\mathbb{S}) \\
\Downarrow \\
K_1(\mathrm{FinSet}) \cong \pi_1^s(\mathbb{S}) \\
\Downarrow \\
\text{the loop } h : \varnothing \sim \{+,-\} \sim \{-,+\} \sim \varnothing.
\end{gathered}$$

This is the clean mathematical sense in which the arithmetic-looking line is the Hopf map in disguise.

§1.12 Cobordism: the movie becomes a circle

The slides then place the motion in three-dimensional space. As time evolves, the moving point configuration sweeps out a one-dimensional manifold. The creation of the pair, their swap, and their annihilation form a closed loop. Because the points carry signs or framings, the resulting circle is a framed one-manifold.

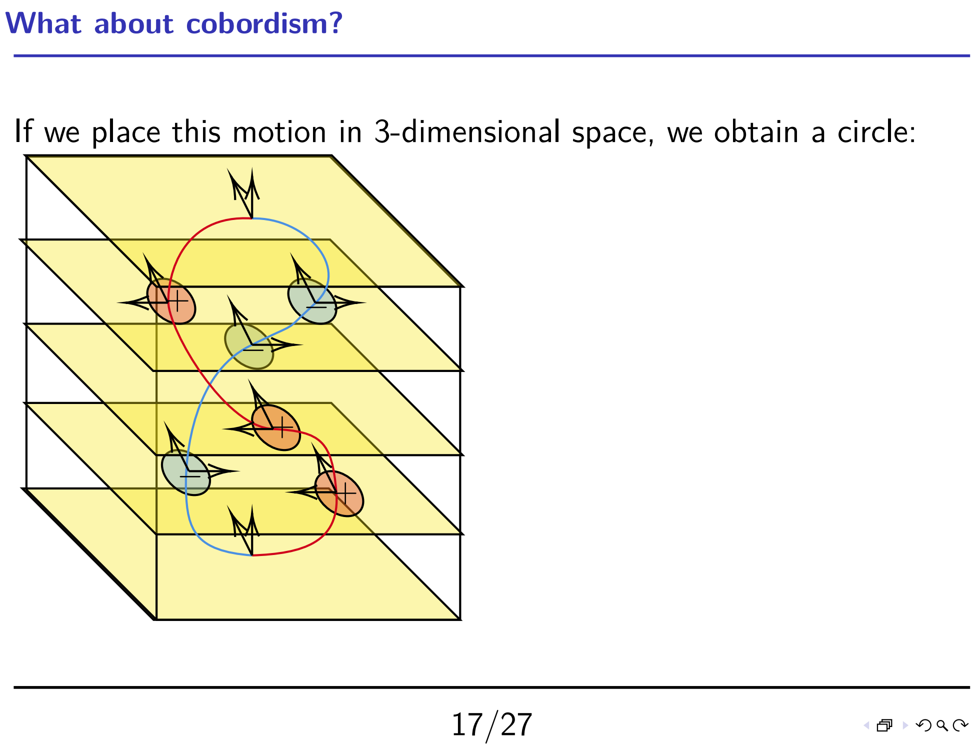

Figure 1.3: From the lecture slides: the creation–swap–annihilation movie becomes a framed circle.

Framed cobordism asks whether this framed circle is the boundary of a framed disk. The answer is no for the loop h. This is another expression of the same nontrivial $\mathbb{Z}/2$-class.

Definition 1.12.1 (Framed cobordism, informal)**.** A framed n-manifold is an n-manifold equipped with a trivialization of its normal bundle. Two framed manifolds are framed cobordant if they form the boundary of a framed manifold one dimension higher.

This is the geometric face of the equation. The path starts and ends at the empty configuration, but its trace is not the boundary of a trivial framed disk. The loop is geometrically nonzero.

§1.13 Algebraic K-theory of a field

After finite sets, the lecture repeats the construction with finite-dimensional vector spaces over a field k. Let

$$\mathcal{V}_k = \mathrm{FinVect}_k/\sim$$

be the moduli space of finite-dimensional vector spaces. Under direct sum, it behaves like a categorified version of addition.

The low K-groups are familiar:

$$K_0(k) \cong \mathbb{Z}, \qquad K_1(k) \cong k^\times.$$

The first says that a vector space is measured by its dimension after group completion. The second says that the determinant detects the abelianized automorphism information:

$$\mathrm{GL}_\infty(k)^{\mathrm{ab}} \cong k^\times.$$

This parallels the finite-set case:

object	K_0	K_1
FinSet	$\mathbb{Z}$	$\mathbb{Z}/2$
$\mathrm{FinVect}_k$	$\mathbb{Z}$	$k^\times$

The lecture now asks for a geometric model of these algebraic K-groups beyond degree one.

§1.14 Intrinsic skein-lasagna

The advanced second half of the slide deck introduces intrinsic skein-lasagna. This is not merely decorative. It is a proposed geometric language for K-groups analogous to how framed cobordism gives a geometric language for stable homotopy.

The first definition is an n-treefold: a space locally homeomorphic to

$$T \times \mathbb{R}^{n-1},$$

where T is a tree. The branching models a controlled kind of singularity.

Given a symmetric monoidal category $(\mathcal{C}, \otimes)$, an n-$(\mathcal{C}, \otimes)$-lasagna is, roughly, a framed n-treefold with labels from $\mathcal{C}$, satisfying compatibility at the branching loci.

Definition 1.14.1 (Intrinsic skein-lasagna, informal)**.** An intrinsic lasagna is a geometric object built from treefold-like pieces, with labels coming from a symmetric monoidal category. The word "intrinsic" means that the object is studied on its own, rather than as something embedded in a previously chosen ambient manifold.

The slide illustrates n-treefolds as spaces locally modelled on $T \times \mathbb{R}^{n-1}$, where T is a tree. For the notebook, the useful point is that branching is allowed but controlled.

The word "intrinsic" is important. Classical skein-lasagna modules often involve embedded objects inside an ambient manifold. Here the lecture emphasizes that one can study the geometry of the lasagna itself, without first choosing an ambient space. If desired, one may imagine the ambient space as $\mathbb{R}^\infty$, but it is no longer part of the primary data.

§1.15 The $(\mathcal{V}_k, \oplus)$-cobordism group

The lecture specializes the lasagna story to

$$(\mathcal{V}_k, \oplus),$$

where $\mathcal{V}_k$ is the category or moduli object of finite-dimensional vector spaces under direct sum. One obtains groups denoted

$$\Omega_n^{\mathcal{V}_k, \oplus}.$$

The examples in low degree match algebraic K-theory:

$$\Omega_0^{\mathcal{V}_k, \oplus} \cong K_0(k) \cong \mathbb{Z},$$

$$\Omega_1^{\mathcal{V}_k, \oplus} \cong K_1(k) \cong k^\times.$$

Remark 1.15.1 — The slide summarizes the first two $(\mathcal{V}_k, \oplus)$-cobordism groups: degree 0 recovers dimension, and degree 1 recovers determinant. In formulas,

$$\Omega_0^{\mathcal{V}_k,\oplus} \cong K_0(k) \cong \mathbb{Z}, \qquad \Omega_1^{\mathcal{V}_k,\oplus} \cong K_1(k) \cong k^\times.$$

This is why vector-space-labelled circles are the K_1 analogue of the finite-set swap loop.

The degree-one case is especially helpful. A circle labelled by an automorphism $f \in \mathrm{GL}(V)$ represents a class, and the determinant gives its value in $k^\times$. A commutator becomes null-cobordant, matching the abelianization of $\mathrm{GL}_\infty(k)$.

This is the lasagna analogue of the finite-set story. In finite sets, a swap gives the nontrivial element of $\mathbb{Z}/2$. In vector spaces, an automorphism gives an element of $k^\times$.

§1.16 Generalized Pontryagin–Thom

The slide deck then states the central claim:

$$K_n(k) \xrightarrow{\cong} \Omega_n^{\mathcal{V}_k,\oplus} \qquad (n \geq 0).$$

This is presented as a generalized Pontryagin–Thom isomorphism.

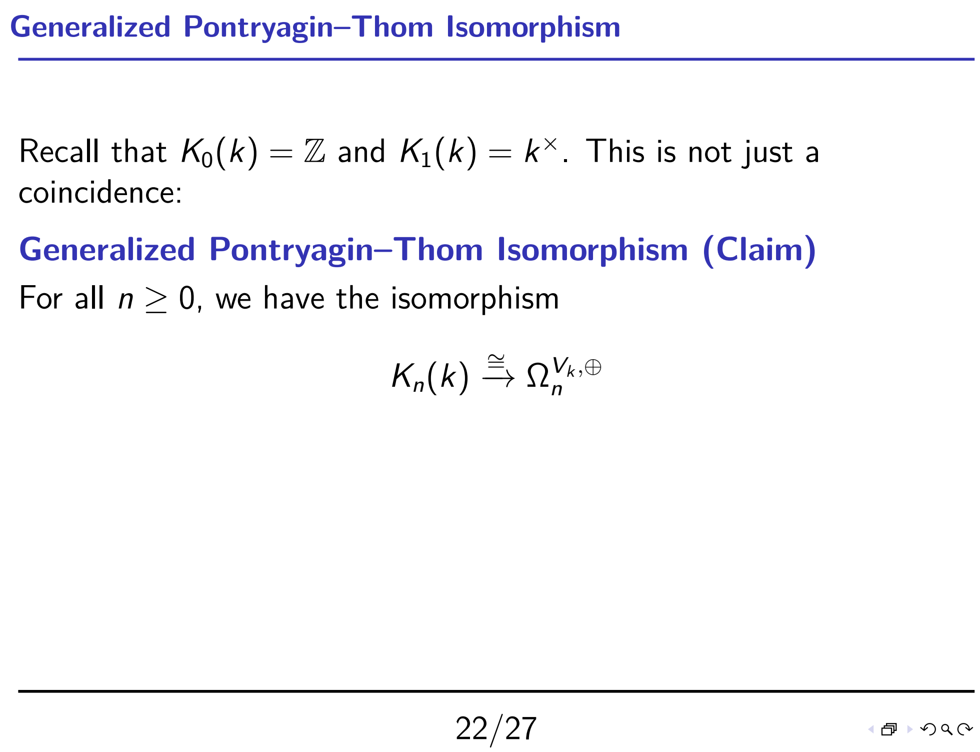

Figure 1.4: From the lecture slides: the generalized Pontryagin–Thom isomorphism identifies $K_n(k)$ with a lasagna-cobordism group.

The analogy is:

classical/stable homotopy	algebraic K-theory side
$\pi_n^s(\mathbb{S})$	$K_n(k)$
Ω_n^{fr}	$\Omega_n^{\mathcal{V}_k,\oplus}$
framed manifolds	vector-space-labelled lasagnas

This is why the lecture moves from elementary arithmetic to Hopf fibration and then to intrinsic skein-lasagna. It is building a geometric model of algebraic K-theory.

§1.17 The K_2 slide and Steinberg relations

In degree two, the lecture recalls the classical presentation

$$K_2(k) \cong k^\times \otimes k^\times / \langle a \otimes (1-a) \rangle.$$

It then considers matrices $f, g \in \mathrm{GL}(k)$ with commutator

$$[f, g] = 1.$$

Such a commuting pair defines a two-dimensional lasagna $C(f, g)$.

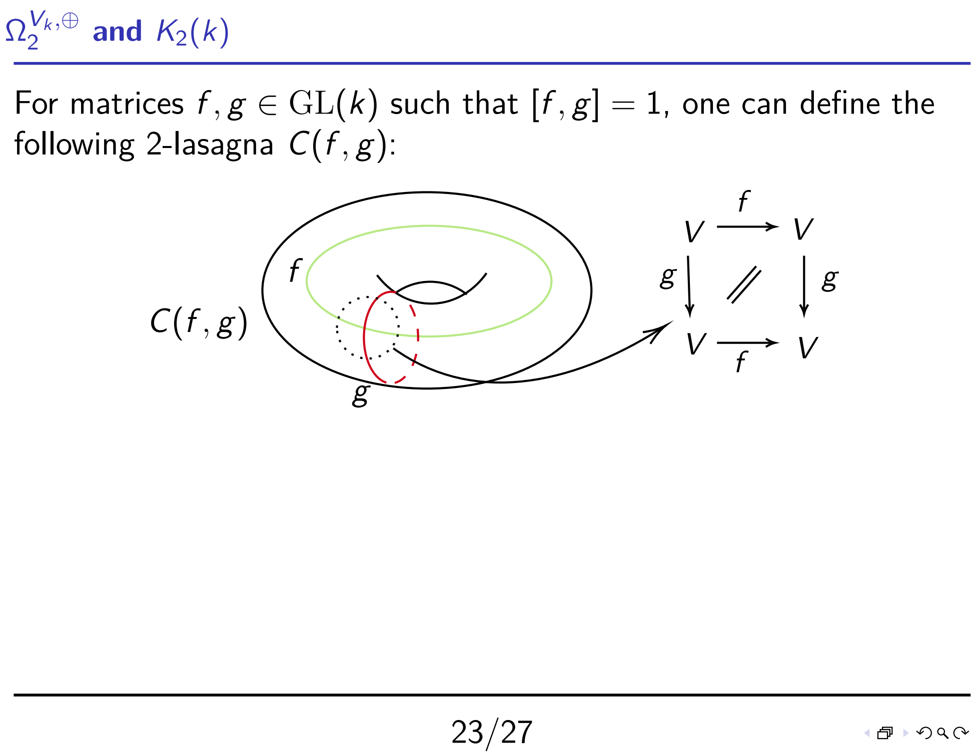

Figure 1.5: From the lecture slides: a two-lasagna $C(f, g)$ associated to a commuting pair.

The next slide gives a more concrete example using elementary matrices. For $i \neq j$, let

$$e_{ij}(a)$$

be the elementary matrix with a in the (i, j)-entry and ones on the diagonal. These matrices satisfy Steinberg relations, such as

$$[e_{ij}(a), e_{kl}(b)] = \begin{cases} e_{il}(ab), & j = k, \\ \mathbf{1}, & \text{otherwise.} \end{cases}$$

The slide then uses Hall–Witt-type commutator identities to produce a nontrivial proof that a certain commutator is the identity. This proof is not just algebraic decoration; it is a blueprint for constructing a bounding three-lasagna.

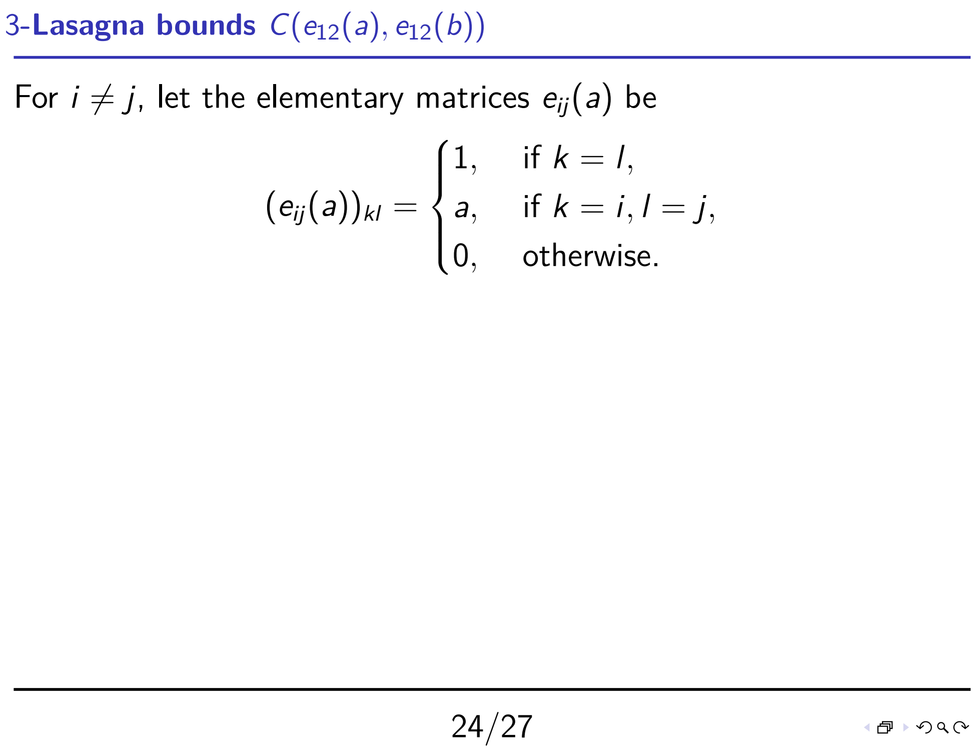

Figure 1.6: From the lecture slides: elementary matrices, Steinberg relations, and the algebraic pattern behind the three-lasagna.

§1.18 The final lasagna diagrams

The final nontrivial slides display a Heegaard splitting and a boundary relation. They are not necessary for understanding the original meme, but they are essential for understanding where the lecture is going. The creation–swap–annihilation loop was the baby example. The lasagna diagrams are the research-level continuation of the same philosophy: algebraic K-theory classes are represented by geometric/cobordism-like objects.

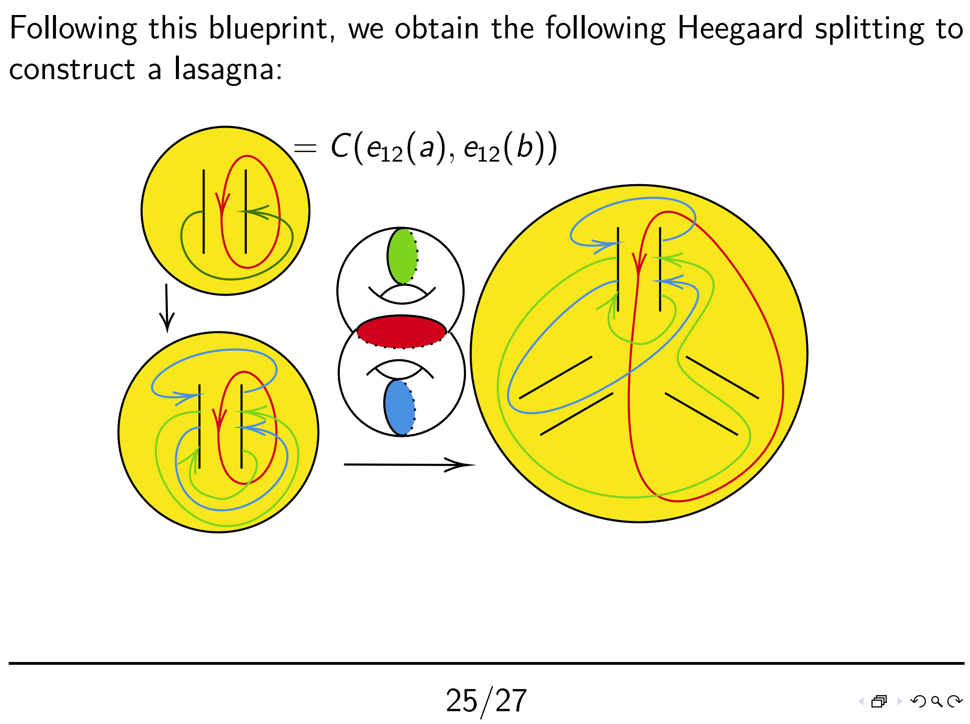

Figure 1.7: From the lecture slides: a Heegaard-splitting picture used to construct a three-lasagna.

At this point the lecture has travelled very far from the original equation. But the path is coherent:

$$\begin{gathered}
\text{equation as path} \\
\Downarrow \\
\text{finite signed configurations} \\
\Downarrow \\
K\text{-theory of finite sets} \\
\Downarrow \\
\text{stable homotopy and framed cobordism} \\
\Downarrow \\
\text{algebraic } K\text{-theory of fields} \\
\Downarrow \\
\text{intrinsic skein-lasagna as a geometric model.}
\end{gathered}$$

§1.19 What the equation finally says

The equation

$$0 = 1 - 1 = -1 + 1 = 0$$

should be read as the name of a loop:

$$h : \varnothing \to \{+, -\} \to \{-, +\} \to \varnothing.$$

It is a loop at zero in the K-space of finite sets. It is not the identity loop. It represents the generator of

$$K_1(\mathrm{FinSet}) \cong \mathbb{Z}/2.$$

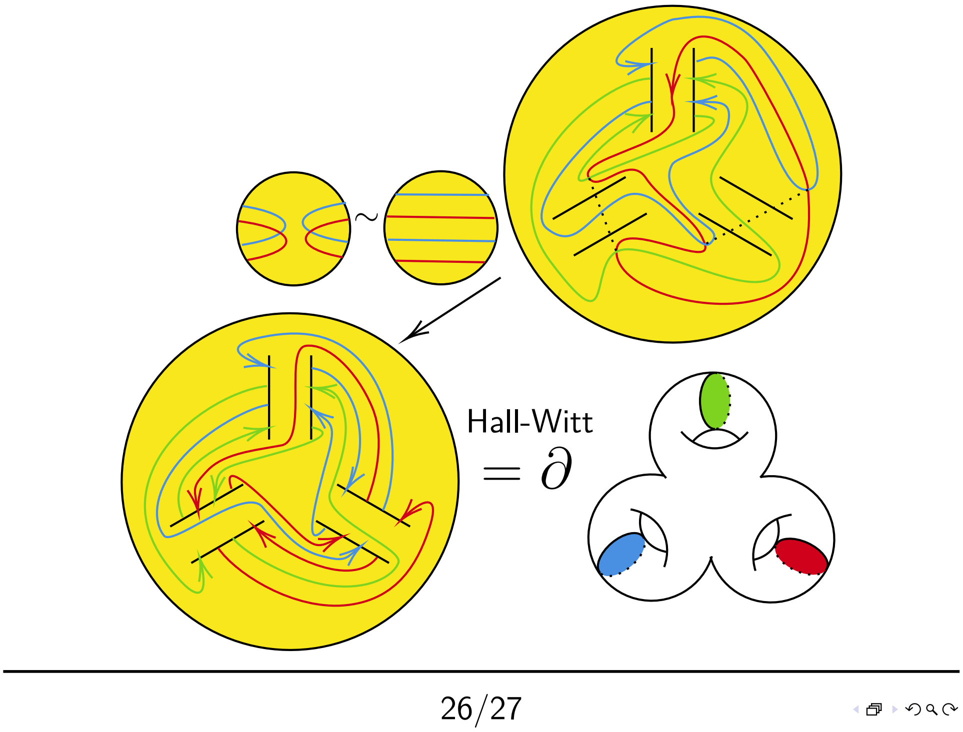

Figure 1.8: From the lecture slides: the boundary relation for the three-lasagna construction.

By Barratt–Priddy–Quillen, this group is

$$\pi_1^s(\mathbb{S}) \cong \mathbb{Z}/2,$$

and h corresponds to the stable Hopf class η.

This is why the Hopf fibration belongs in the same story as a signed finite-set calculation. The shared object is not an ordinary number. It is a stable homotopy class.

§1.20 Structural viewpoint

The two notes should now be read in chronological order rather than as if the first one already knew the second one's answer. The earlier note preserves the first route: a summer memory of the Hopf fibration, a lecture poster, a meme, and the attempt to explain the line through analytic summation, infinite cancellation, and the Eilenberg swindle. This second note records the later correction: the same vocabulary of K-theory and stability is present, but the mathematical emphasis moves from vanishing to a nontrivial finite loop.

The hierarchy is:

first reaction	later understanding
$0 = 0$	a loop from 0 to 0
$1 - 1 + 1 - 1 + \cdots$	one created $+/-$ pair
Cesàro/Abel/eta	finite configuration space
Eilenberg swindle	group completion of finite sets
infinite absorption	creation–swap–annihilation movie
vanishing of invariants	$K_1(\mathrm{FinSet}) \cong \mathbb{Z}/2$
triviality of the infinite	nontriviality of the finite
arithmetic/regularization instinct	stable Hopf class η

This distinction is the real lesson of the reflection. The first idea was not stupid; it was a reasonable search direction because K-theory really does interact with group completion, stability, and swindle arguments. But it placed the weight on the wrong side. It tried to make the equation true by allowing infinitely flexible cancellation. The lecture instead makes the equation interesting by refusing to collapse the finite movie to its endpoint.

The advanced tail of the lecture should also be remembered. The same style of thought continues beyond finite sets. For a field k, algebraic K-groups can be approached through vector-space-labelled lasagnas:

$$K_n(k) \cong \Omega_n^{\mathcal{V}_k,\oplus}.$$

The final slides are not random extra material. They show that the playful equation is the doorway into a geometric model of algebraic K-theory.

So the final lesson is not that 0 is secretly nonzero, and it is not that infinity makes all cancellation meaningless. The lesson is sharper: in higher mathematics, the space of ways for 0 to be 0 can contain a nontrivial loop, and in this example that loop is already visible in the smallest possible finite signed configuration.