Green's Theorem, Conservative Fields, and the Planar Stokes Principle

N-20260417

§1.1 Green's theorem

Green's theorem relates circulation around a boundary to local rotation inside the region.

Theorem 1.1.1 (Green's theorem)
Let $D \subseteq \mathbb{R}^2$ be a positively oriented region with piecewise smooth simple closed boundary $C = \partial D$. If P and Q have continuous first partial derivatives on a neighborhood of D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

The left side is circulation along the boundary. The right side is total scalar curl over the region.

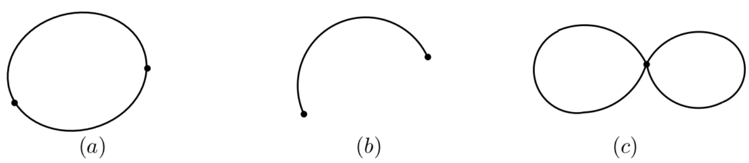

Figure 1.1: Simple closed curves and non-simple closed curves in the statement of Green's theorem.

§1.2 Proof idea: rectangles and cancellation

For a rectangle $[a, b] \times [c, d]$, the theorem follows from the fundamental theorem of calculus. The $Q\,dy$ part detects horizontal change in Q, and the $P\,dx$ part detects vertical change in P with the opposite sign.

For a region subdivided into many rectangles, internal edges are traversed twice in opposite directions. Their line integrals cancel. Only the outer boundary remains.

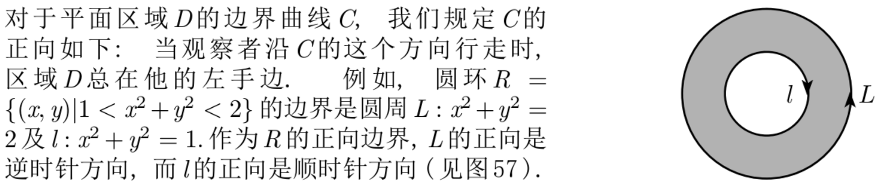
对于平面区域D的边界曲线C，我们规定C的正向如下：当观察者沿C的这个方向行走时，区域D总在他的左手边。例如，圆环$R = \{(x, y) | 1 < x^2 + y^2 < 2\}$的边界是圆周$L: x^2 + y^2 = 2$及$l: x^2 + y^2 = 1$. 作为$R$的正向边界，$L$的正向是逆时针方向，而$l$的正向是顺时针方向（见图57）.

Figure 1.2: For a vertically simple region, Green's theorem is proved by reducing to one-variable fundamental theorem computations.

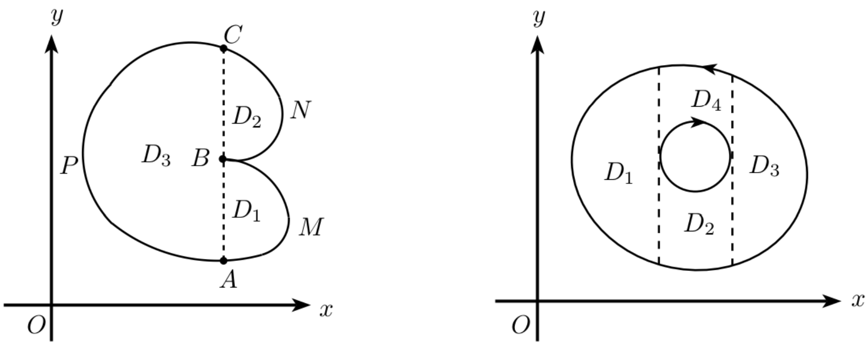

Figure 1.3: Subdivision by auxiliary arcs: internal boundary contributions cancel, leaving only the outer boundary.

§1.3 Conservative fields and path independence

A vector field $F = (P, Q)$ is **conservative** on a region U if there is a potential function ϕ such that

$$\nabla\phi = F.$$

Then

$$\int_C P\,dx + Q\,dy = \phi(B) - \phi(A)$$

for every path C from A to B. Thus the integral depends only on endpoints.

Proposition 1.3.1
If $F = \nabla\phi$, then $P_y = Q_x$.

Proof. If $P = \phi_x$ and $Q = \phi_y$, then under the usual continuity hypotheses,

$$P_y = \phi_{xy} = \phi_{yx} = Q_x.$$

□

The converse requires a topological hypothesis.

Theorem 1.3.2 (Conservative-field test)
If $U \subseteq \mathbb{R}^2$ is simply connected and $P_y = Q_x$ on U, then $F = (P, Q)$ is conservative on U.

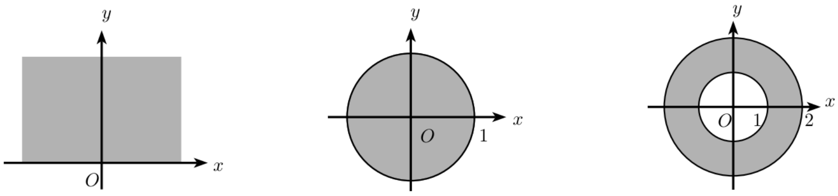

Figure 1.4: Simply connected and multiply connected plane regions. The distinction matters for path independence and conservative fields.

§1.4 The punctured-plane warning

The standard example is

$$F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

on $\mathbb{R}^2 \setminus \{0\}$. A direct computation gives $P_y = Q_x$ away from the origin, but around the unit circle $r(t) = (\cos t, \sin t)$,

$$F(r(t)) = (-\sin t, \cos t), \qquad r'(t) = (-\sin t, \cos t),$$

so

$$\oint_{S^1} F \cdot dr = \int_0^{2\pi} 1\, dt = 2\pi.$$

Therefore no global potential exists on the punctured plane. The hole is detected by circulation.

§1.5 Area by Green's theorem

Green's theorem can compute area. Choose

$$P = -\frac{y}{2}, \qquad Q = \frac{x}{2}.$$

Then $Q_x - P_y = 1$, so

$$\operatorname{area}(D) = \iint_D 1\, dA = \frac{1}{2}\oint_{\partial D} x\, dy - y\, dx.$$

Example 1.5.1

For the unit circle $x = \cos t$, $y = \sin t$,

$$\frac{1}{2}\int_0^{2\pi} (xy' - yx')\, dt = \frac{1}{2}\int_0^{2\pi} 1\, dt = \pi.$$

§1.6 Differential forms notation

Let

$$\omega = P\, dx + Q\, dy.$$

Then

$$d\omega = (Q_x - P_y)\, dx \wedge dy.$$

Green's theorem becomes

$$\int_{\partial D} \omega = \int_D d\omega.$$

This is Stokes' theorem in the plane.

The algebraic slogan is: integrate a form over the boundary, or integrate its derivative over the interior.

§1.7 Closed forms, exact forms, and topology

A one-form ω is **closed** if $d\omega = 0$. It is **exact** if $\omega = d\phi$ for some function ϕ. Exact forms are always closed, because $d^2 = 0$. The converse depends on the domain.

On simply connected plane regions, closed one-forms are exact. On the punctured plane, the form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

is closed but not exact, because

$$\int_{S^1} \omega = 2\pi.$$

§1.8 Connection with de Rham cohomology

The first de Rham cohomology group is

$$H^1_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^1(M) \to \Omega^2(M))}{\mathrm{im}(d : \Omega^0(M) \to \Omega^1(M))}.$$

It measures closed one-forms modulo exact one-forms. In elementary vector calculus language, it measures the obstruction to finding potentials for curl-free fields.

§1.9 Winding number hidden in the punctured-plane example

For the one-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2},$$

write $x = r\cos\theta$, $y = r\sin\theta$. Then formally

$$\omega = d\theta.$$

The word "formally" matters: θ is not a globally single-valued function on $\mathbb{R}^2 \setminus \{0\}$. Around a closed loop γ,

$$\frac{1}{2\pi} \int_\gamma \omega$$

is the winding number of γ about the origin.

For the unit circle, the winding number is 1. For a loop going twice around, it is 2. For a loop not enclosing the origin, it is 0. Thus the line integral is not merely detecting that a potential is missing; it measures how the curve winds around the hole.

§1.10 From Green's theorem to residues

The same punctured-plane form is connected to complex analysis. Since

$$\frac{dz}{z} = \frac{dx + i\,dy}{x + iy},$$

one computes

$$\operatorname{Im}\frac{dz}{z} = \frac{x\,dy - y\,dx}{x^2 + y^2} = \omega.$$

Thus

$$\int_{S^1} \omega = 2\pi$$

is the imaginary part of

$$\int_{S^1} \frac{dz}{z} = 2\pi i.$$

The residue theorem is a complex-analytic refinement of the same topological fact: a closed integral can detect a puncture and its winding number.

§1.11 Cohomology as bookkeeping for failed potentials

The statement "curl-free implies conservative" is not universally true; it is true when the relevant cohomology vanishes. In de Rham notation,

$$H^1_{\mathrm{dR}}(M) = \frac{\text{closed one-forms}}{\text{exact one-forms}}.$$

A nonzero class is represented by a closed form that is not the differential of any global function.

For $M = \mathbb{R}^2 \setminus \{0\}$, the class of ω above generates $H^1_{\mathrm{dR}}(M) \cong \mathbb{R}$. The integral over S^1 is a period pairing:

$$([\omega], [S^1]) \longmapsto \int_{S^1} \omega.$$

So the elementary warning about holes is really the first example of pairing cohomology with cycles.

§1.12 Boundary cancellation and the algebra $\partial^2 = 0$

The rectangle-subdivision proof of Green's theorem says that every internal edge appears twice with opposite orientations. Algebraically this is the principle

$$\partial^2 = 0: \quad \text{the boundary of a boundary cancels.}$$

De Rham theory has the dual identity

$$d^2 = 0.$$

Stokes' theorem connects them:

$$\int_{\partial C} \omega = \int_C d\omega.$$

If one applies this again, both sides become zero for structural reasons. Thus Green's theorem is not an isolated vector-calculus trick; it is the two-dimensional visible face of a chain/cochain duality.

§1.13 What Green's theorem checks in practice

Green's theorem converts a boundary integral into an area integral:

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

In a computation this requires more than recognizing the formula. The boundary must be positively oriented, the vector field must be defined on a region containing D, and holes in the domain must be accounted for by extra boundary components with the correct orientation.

The punctured-plane example explains why this check matters. The form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

is closed away from the origin, but its integral around S^1 is nonzero, so it is not globally exact on $\mathbb{R}^2 \setminus \{0\}$. Green's theorem, the identity $d^2 = 0$, and the cancellation of internal boundaries are therefore three views of the same mechanism: local derivative data controls boundary circulation only after the topology of the region has been included.