Generalized Inverses, Projections, and Least-Squares Geometry

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§1.1 Why ordinary inverse is not enough

For $A \in \mathbb{C}^{m\times n}$, the system $Ax = b$ may be overdetermined, underdetermined, singular, or rank deficient. The Moore-Penrose inverse A^+ gives a unified optimal answer.

§1.2 Four regimes

I	$m = n$	$r = n$	ordinary inverse
II	$m > n$	$r = n$	least squares
III	$m < n$	$r = m$	minimum norm
IV	any	$r < \min(m, n)$	rank deficient

§1.3 Penrose equations

The Moore-Penrose inverse is the unique X satisfying

$$AXA = A, \quad XAX = X, \quad (AX)^H = AX, \quad (XA)^H = XA.$$

It is denoted A^+.

§1.4 Fifteen generalized inverses

Choosing any nonempty subset of the four Penrose equations gives one of 15 generalized-inverse classes. Only the Moore-Penrose inverse is unique for every matrix.

§1.5 Projection transformations

If $\mathbb{C}^n = L \oplus M$, the projection onto L along M maps $x = y + z$ to y, where $y \in L, z \in M$.

§1.6 Idempotence and orthogonality

A matrix is a projection iff

$$P^2 = P.$$

It is an orthogonal projection iff

$$P^2 = P, \qquad P^H = P.$$

§1.7 Column and row-space projections

AA^+ is the orthogonal projection onto $R(A)$. A^+A is the orthogonal projection onto $R(A^H)$.

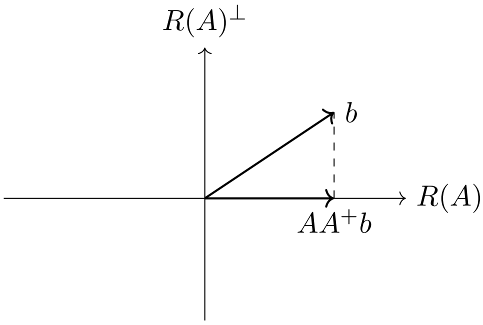

Figure 1.1: The vector AA^+b is the orthogonal projection of b onto the column space.

§1.8 SVD computation

If

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^H,$$

then

$$A^+ = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^H.$$

Only nonzero singular values are inverted.

§1.9 Full-rank computation

If $A = FG$ is a full-rank decomposition, then

$$A^+ = G^H(GG^H)^{-1}(F^HF)^{-1}F^H.$$

§1.10 Greville recursion

Greville's method updates A_k^+ when a new column is appended. It is useful for column-by-column or online data, although numerical stability still requires care.

§1.11 Iterative approximations

Iterative methods for A^+ often require a step size satisfying a condition such as

$$0 < \alpha < \frac{2}{\sigma_{\max}^2}.$$

Convergence is again a spectral-radius issue.

§1.12 Solvability

The system $Ax = b$ is solvable iff
$$AA^+b = b.$$
If it is solvable, the general solution is
$$x = A^+b + (I - A^+A)y.$$

§1.13 Least squares and minimum norm

For every b, A^+b is the least-squares solution minimizing $\|Ax - b\|_2$. Among all minimizers, it has minimum 2-norm.

§1.14 Identities

$$(A^+)^+ = A, \qquad (A^H)^+ = (A^+)^H, \qquad \operatorname{rank} A^+ = \operatorname{rank} A.$$
In general, $(AB)^+ \neq B^+A^+$ without additional assumptions.

§1.15 Reading the Moore–Penrose inverse geometrically

expression	geometric meaning	use
A^+	inverse on nonzero singular directions	canonical generalized inverse
AA^+	orthogonal projection onto $R(A)$	test the reachable part of b
A^+A	orthogonal projection onto $R(A^H)$	remove the null-space ambiguity
$AA^+b = b$	$b \in R(A)$	exact solvability
A^+b	minimum-norm least-squares vector	canonical approximate solution

The SVD formula makes the construction explicit: invert the nonzero singular values and leave the zero singular directions untouched. If b is in the column space, A^+b solves $Ax = b$ and chooses the solution orthogonal to $N(A)$. If b is not in the column space, AA^+b is the closest reachable vector in $R(A)$, so A^+b solves the least-squares problem with minimum norm among all least-squares solutions. This also explains why $(AB)^+ = B^+A^+$ is not generally safe: the projection geometry of the two factors need not line up.