Numerical Series, Power Series, Taylor Series, and Improper Integrals

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§1.1 Why these topics belong together

The long Markdown note moves from numerical series to function series, then to power series and improper integrals. The common theme is convergence: when does an infinite summation or an infinite accumulation define a finite object, and when may we manipulate it like a finite expression?

§1.2 Numerical series

A numerical series is

$$\sum_{n=1}^{\infty} a_n, \qquad s_N = \sum_{n=1}^{N} a_n.$$

It converges if the partial sums s_N converge. The first necessary test is

$$a_n \to 0.$$

If this fails, the series diverges. If it holds, nothing is decided.

The geometric series is the base model:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad |r| < 1.$$

The harmonic series shows why terms tending to zero is insufficient:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

A standard proof groups terms:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots .$$

Each block after the first contributes at least $1/2$.

§1.3 Positive-term series

For $a_n \geq 0$, convergence is controlled by size.

Comparison test: if $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Limit comparison: if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \qquad 0 < c < \infty,$$

then $\sum a_n$ and $\sum b_n$ have the same convergence behavior.

The p-series is the main comparison model:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges iff $p > 1$.

The examples in the original note include

$$\sum \sin \frac{1}{n}, \qquad \sum \ln\left(1 + \frac{1}{n^2}\right), \qquad \sum \left(\sqrt{n+1} - \sqrt{n}\right).$$

The first behaves like $\sum 1/n$ and diverges. The second behaves like $\sum 1/n^2$ and converges. The third behaves like $\sum 1/(2\sqrt{n})$ and diverges.

§1.4 Ratio and root tests

The ratio test is suited to factorials and exponentials:

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1$, the series converges absolutely; if $L > 1$, it diverges; if $L = 1$, the test is inconclusive.

The root test is suited to expressions of the form $(\cdots)^n$:

$$L = \limsup_{n \to \infty} \sqrt[n]{|a_n|}.$$

Again $L < 1$ gives absolute convergence, $L > 1$ gives divergence, and $L = 1$ is inconclusive.

Typical original examples include $\sum a^n/n^s$, factorials in the numerator or denominator, and powers with variable bases. The rule is not to memorize which test belongs to which problem, but to inspect which part of a_n changes fastest.

§1.5 Alternating and arbitrary-sign series

Absolute convergence means

$$\sum |a_n| < \infty.$$

Conditional convergence means $\sum a_n$ converges but $\sum |a_n|$ diverges.

The alternating-series test says that if $b_n \downarrow 0$, then

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

converges. The alternating harmonic series is the standard example.

Dirichlet's test: if partial sums $A_N = \sum_{n=1}^{N} a_n$ are bounded and $b_n \downarrow 0$, then $\sum a_n b_n$ converges. This handles examples such as

$$\sum_{n=1}^{\infty} \frac{\cos n}{n}.$$

Abel's test is a neighboring result where one factor has a convergent series and the other is monotone bounded.

§1.6 Function series and uniform convergence

A function series is

$$\sum_{n=1}^{\infty} u_n(x).$$

Pointwise convergence asks for each fixed x. Uniform convergence asks whether the tail becomes small simultaneously for all x in a set:

$$\sup_{x\in E} |s_N(x) - s(x)| \to 0.$$

Uniform convergence is what allows continuity and integration to pass to the limit. Without it, a limit of continuous functions may fail to be continuous.

The Weierstrass M-test says that if

$$|u_n(x)| \le M_n, \qquad \sum M_n < \infty,$$

then $\sum u_n$ converges uniformly and absolutely.

§1.7 Power series

A power series centered at x_0 is

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

There exists a radius of convergence R such that the series converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$. Endpoints require separate testing.

A common formula is

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

when the limit exists. Missing powers require care: if the series contains $(x - x_0)^{2n}$, one should apply the ratio/root test directly rather than blindly reading a coefficient sequence as though every power were present.

Inside the interval of convergence, power series may be differentiated and integrated term by term:

$$\left(\sum a_n(x - x_0)^n\right)' = \sum n a_n(x - x_0)^{n-1}.$$

The radius of convergence remains the same, though endpoints may change.

§1.8 Taylor series

The Taylor series at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

At $a = 0$, it is a Maclaurin series. Standard expansions include

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

Many exam examples are indirect expansions: rewrite the function in terms of known series, substitute, integrate, differentiate, or multiply series.

§1.9 Improper integrals

Improper integrals appear when the interval is infinite or the integrand is unbounded:

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx,$$

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if f is singular at b.

The two fundamental p-templates are

$$\int_1^\infty \frac{dx}{x^p} \quad \text{converges iff } p>1,$$

$$\int_0^1 \frac{dx}{x^p} \quad \text{converges iff } p<1.$$

Logarithmic refinements include

$$\int_e^\infty \frac{dx}{x(\ln x)^p},$$

which converges iff $p>1$.

For sign-changing improper integrals, absolute convergence means $\int |f| < \infty$. Conditional convergence often comes from oscillation, as in Dirichlet-type integrals.

§1.10 Unified viewpoint

Series and improper integrals ask the same question: can infinitely many small contributions accumulate to a finite value? Positive objects are controlled by size; sign-changing objects are controlled by cancellation plus size. Power series add a new idea: convergence depends on the point x, producing a domain where infinite algebraic manipulation becomes legitimate.

§1.11 Uniform convergence as norm convergence

For a function series $\sum u_n(x)$, uniform convergence on E is convergence in the sup norm:

$$\|s_N - s\|_{\infty,E} = \sup_{x\in E}|s_N(x)-s(x)| \to 0.$$

The Weierstrass M-test is simply comparison in this norm. If $|u_n(x)| \le M_n$ and $\sum M_n < \infty$, then the tails satisfy

$$\sup_x \left|\sum_{n=N}^\infty u_n(x)\right| \le \sum_{n=N}^\infty M_n \to 0.$$

Thus the elementary test is really completeness of a normed function space at work.

§1.12 Analytic functions and convergence radius

A power series

$$\sum a_n(z-z_0)^n$$

defines a holomorphic function inside its disk of convergence. The radius is controlled by the nearest complex singularity, not merely by real-variable behavior. This explains why real Taylor series sometimes stop converging at a point where the real function looks harmless: the obstruction may live off the real axis.

For example,

$$\frac{1}{1+x^2}$$

has real Taylor series at 0

$$1-x^2+x^4-\cdots,$$

with radius 1, because the complex singularities are at $x=\pm i$.

§1.13 Dominated convergence and improper integrals

Improper integrals and parameter-dependent integrals become cleaner in measure language. If $f_n \to f$ pointwise and $|f_n| \le g$ with g integrable, dominated convergence gives

$$\int f_n \to \int f.$$

This justifies many limiting operations that elementary calculus handles case by case.

A useful contrast is between absolute convergence and conditional cancellation. The integral

$$\int_1^\infty \frac{\sin x}{x}\,dx$$

converges conditionally by oscillation, but

$$\int_1^\infty \left|\frac{\sin x}{x}\right|\,dx$$

diverges. This is the integral analogue of conditional series.

§1.14 Tauberian flavor

Power series also encode sequences. Given a_n, define

$$F(r)=\sum_{n=0}^{\infty} a_n r^n, \qquad 0<r<1.$$

Abelian theorems say that ordinary convergence of $\sum a_n$ often implies limiting behavior of $F(r)$ as $r \to 1^-$. Tauberian theorems ask for converses under additional hypotheses. This is the research-level continuation of the elementary question: how does convergence of coefficients relate to behavior of the generating function near the boundary?