The Second-Order Wave Equation and d'Alembert's Formula

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§1.1 The equation and its physical meaning

The one-dimensional wave equation is

$$u_{tt} = c^2 u_{xx},$$

where $u(x,t)$ is the vertical displacement of a string and $c > 0$ is the wave speed. The equation says that acceleration is proportional to curvature. This is physically reasonable: translating the whole string upward produces no restoring force, and a straight slanted string has no net restoring force; only bending produces acceleration.

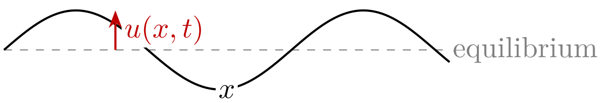

Figure 1.1: A vibrating string. The vertical displacement $u(x,t)$ is measured from the equilibrium line.

The equation is called hyperbolic because its highest-order part has the sign pattern of $\partial_t^2 - c^2\partial_x^2$. This name is about the algebraic type of the differential operator, not about hyperbolas as curves.

§1.2 Factorization into transport operators

The wave operator factors as

$$\partial_t^2 - c^2\partial_x^2 = (\partial_t + c\partial_x)(\partial_t - c\partial_x).$$

Since the two constant-coefficient operators commute, the wave equation can be read as a pair of first-order transport equations. Any function of $x + ct$ solves the equation, and any function of $x - ct$ also solves it.

Theorem 1.2.1 (d'Alembert form)

Every twice differentiable solution of

$$u_{tt} = c^2 u_{xx}$$

on a simply connected region in the (x,t)-plane can locally be written as

$$u(x,t) = F(x + ct) + G(x - ct).$$

The term $F(x + ct)$ travels left, and $G(x - ct)$ travels right.

Proof. Set
$$\xi = x + ct, \qquad \eta = x - ct.$$
By the chain rule,
$$\partial_x = \partial_\xi + \partial_\eta, \qquad \partial_t = c\partial_\xi - c\partial_\eta.$$
A direct calculation gives
$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}.$$
Thus the wave equation is $u_{\xi\eta} = 0$. Integrating once in η gives $u_\xi = A(\xi)$, and integrating in ξ gives $u = F(\xi) + G(\eta)$. □

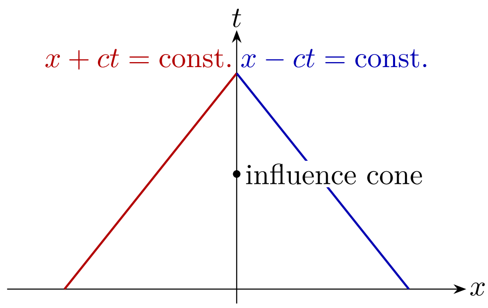

Figure 1.2: The two characteristic families of the wave equation. A point receives information from both left-moving and right-moving characteristic directions.

§1.3 Initial-value problem on the whole line

Suppose
$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x).$$
From $u = F(x + ct) + G(x - ct)$, the first condition gives
$$F(x) + G(x) = \phi(x).$$
The second condition gives
$$cF'(x) - cG'(x) = \psi(x).$$
Solving these two equations yields the d'Alembert formula
$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$
This formula shows finite propagation speed: the value at (x, t) depends only on the initial data in the interval $[x - ct, x + ct]$.

Example 1.3.1 (Zero initial velocity)
Let
$$\phi(x) = \frac{1}{1 + x^2}, \qquad \psi(x) = 0.$$
Then
$$u(x, t) = \frac{1}{2}\left[\frac{1}{1 + (x + ct)^2} + \frac{1}{1 + (x - ct)^2}\right].$$
The initial bump splits into two half-sized bumps travelling in opposite directions.

§1.4 Finite strings and Fourier series

For a string fixed at $x = 0$ and $x = L$, the boundary conditions are

$$u(0, t) = u(L, t) = 0.$$

The spatial eigenfunctions satisfying these boundary conditions are

$$\sin \frac{n\pi x}{L}, \qquad n = 1, 2, 3, \dots.$$

Separation of variables gives normal modes

$$u_n(x, t) = \sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right).$$

A general solution is a Fourier sine series built from these modes:

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right).$$

This is the first bridge from the Fourier-series note to PDEs: boundary conditions choose the spatial basis, while the PDE determines the time oscillation of each coefficient.

§1.5 Energy conservation

The wave equation has a conserved energy. On the whole line, assume u and its derivatives decay sufficiently fast as $|x| \to \infty$. Define

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left(u_t(x, t)^2 + c^2 u_x(x, t)^2 \right) dx.$$

Then

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{xt}) \, dx.$$

Using $u_{tt} = c^2 u_{xx}$,

$$\frac{dE}{dt} = c^2 \int_{-\infty}^{\infty} (u_t u_{xx} + u_x u_{xt}) \, dx.$$

Integrating the first term by parts gives

$$\int u_t u_{xx} \, dx = -\int u_{tx} u_x \, dx,$$

so the two terms cancel and $E'(t) = 0$.

Theorem 1.5.1 (Uniqueness by energy)

For the wave equation on the whole line with sufficiently decaying smooth solutions, the initial data $u(x, 0)$ and $u_t(x, 0)$ determine at most one solution.

Proof. If u and v are two solutions with the same initial data, set $w = u - v$. Then w solves the homogeneous wave equation with zero initial displacement and zero initial velocity. Its energy at $t = 0$ is zero. Since energy is conserved, $E_w(t) = 0$ for all t. Hence $w_t = 0$ and $w_x = 0$, so w is constant in space and time. The initial data give $w = 0$. □

Energy is often more robust than an explicit formula. On domains where d'Alembert's formula is unavailable, energy still gives uniqueness and stability.

§1.6 Inhomogeneous wave equation and Duhamel's principle

Consider

$$u_{tt} - c^2 u_{xx} = F(x,t), \qquad u(x,0) = 0, \qquad u_t(x,0) = 0.$$

The forcing F can be viewed as continuously injecting infinitesimal initial velocities. Duhamel's principle gives

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(\xi, s)\, d\xi\, ds.$$

This formula is the forced analogue of d'Alembert's formula. The triangular integration region is the past light cone of (x,t).

Example 1.6.1 (Localized forcing)
If F is supported near a point (x_*, s_*), then it can influence (x,t) only if

$$|x - x_*| \le c(t - s_*), \qquad s_* \le t.$$

This is finite propagation speed in another form. Information does not instantly appear everywhere; it travels along the characteristic cone.

§1.7 Boundary conditions on a finite interval

For a finite string $0 < x < L$, the physical boundary condition depends on how the endpoint is constrained.

- Fixed endpoints give Dirichlet conditions:

  $$u(0,t) = u(L,t) = 0.$$

- Free endpoints give Neumann conditions:

  $$u_x(0,t) = u_x(L,t) = 0.$$

- A damped or elastic endpoint can lead to mixed conditions such as

  $$\alpha u + \beta u_x = 0$$

  at an endpoint.

The boundary condition selects the spatial eigenfunctions. Dirichlet conditions produce sine modes, while Neumann conditions produce cosine modes. This explains why half-range Fourier expansions are not only computational tricks: they encode boundary behavior.

§1.8 Deriving the Fourier coefficients

For the fixed string, write

$$u(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L}.$$

Substituting into $u_{tt} = c^2 u_{xx}$ gives

$$q_n''(t) + \left(\frac{n\pi c}{L}\right)^2 q_n(t) = 0.$$

Hence

$$q_n(t) = A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}.$$

If

$$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x),$$

then

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}\,dx,$$

and

$$\frac{n\pi c}{L}B_n = \frac{2}{L}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx.$$

Thus

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx.$$

The role of orthogonality is explicit: it decouples one PDE into infinitely many harmonic oscillators.

§1.9 Standing waves and travelling waves

The functions

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

are standing waves. Their nodes are fixed in space. By contrast, d'Alembert's formula writes the solution as travelling waves. These are not contradictory descriptions. A standing wave is the superposition of a left-moving and a right-moving travelling wave of the same frequency.

For example,

$$\sin(kx)\cos(kct) = \frac{1}{2}\left[\sin(k(x+ct)) + \sin(k(x-ct))\right].$$

The Fourier description is best adapted to bounded intervals and boundary conditions; the travelling-wave description is best adapted to the whole line and propagation.

§1.10 The wave equation with damping

A simple damped string model is

$$u_{tt} + \gamma u_t = c^2 u_{xx}, \qquad \gamma > 0.$$

The energy is no longer conserved. Multiplying by u_t and integrating gives

$$\frac{dE}{dt} = -\gamma\int u_t^2\,dx \le 0.$$

Thus damping removes kinetic energy. This example shows how a small change in the PDE changes the qualitative behavior: the undamped wave equation transports energy, while the damped wave equation dissipates it.